INTRODUCTION TO TOPOLOGICAL K-THEORY

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Time and format: Summer semester 2016, 2×2h of lectures and 1×2h of exercises per week.

Level: Master. PhD students are also welcome!

Subject: A branch of algebraic topology, topological K-theory studies spaces X in terms of abelian groups $K(X)$ and $KO(X)$ called the complex and real K-groups of X, respectively. These groups have direct geometric content: on a compact space X, the group $K(X)$ can be obtained in a simple way out of isomorphism classes of complex vector bundles on X, while $KO(X)$ arises in a similar way from real vector bundles. The K-groups are surprisingly powerful invariants, and they have beautiful applications such as an extremely simple solution to the famous Hopf invariant one problem. One geometric consequence of this problem is that the sphere S^n is parallelizable if and only if $n = 0$, 1, 3 or 7, while another (purely algebraic!) consequence is that $\mathbb{R}^n$ admits the structure of a real division algebra if and only if $n = 1$, 2, 4 or 8. The K-groups are also of some relevance to string theory in theoretical physics, where they show up as homes for D-brane charges.

A striking feature of the K-groups is a periodicity phenomenon called Bott periodicity. For a space X equipped with a basepoint $x_0 \in X$, write ΣX for the suspension of X, so that

$$\Sigma X = S^1 \times X/(S^1 \times \{x_0\} \cup \{1\} \times X).$$

By the Bott periodicity theorem, the reduced complex K-group of a pointed compact space X is isomorphic to that of $\Sigma^2 X$, while the same is true of the reduced real K-groups of X and $\Sigma^8 X$. These results are arguably among the most amazing theorems in algebraic topology. Why should vector bundles on the respective two spaces have such a close connection?

Topics: Vector bundles and their classification, real and complex K-groups, Bott periodicity, Adams operations, solution to the Hopf invariant one problem, applications of the Hopf invariant one problem

Prerequisites: I will try to keep the prerequisites to a minimum. Some point-set topology and algebra will be necessary, but no previous background in algebraic topology will be required.

Literature:

- Michael F. Atiyah, K-theory. Second edition. Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989
- Max Karoubi, K-theory: An Introduction. Reprint of the 1978 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008.
- Allen Hatcher, Vector Bundles and K-theory. Draft for a book, available at `https://www.math.cornell.edu/~hatcher/VBKT/VBpage.html`

Questions? Contact me at `anssi.lahtinen@uni-hamburg.de`