INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 2

April 12, 2016

Problem 1. Let $M \to S^1$ be the Möbius line bundle

$$[0,1] \times \mathbf{R} \,/\, (0,v) \sim (1,-v) \longrightarrow [0,1] \,/\, 0 \sim 1, \quad [(t,v)] \longmapsto [t].$$

Let $f\colon S^1 \to S^1$ be the map $z \mapsto z^2$ (where we now interpret $S^1 \subset \mathbf{C}$). Show that the pullback f^*M is the trivial line bundle.

Problem 2. More generally, what is the pullback of M along the map $S^1 \to S^1$, $z \mapsto z^n$?

Problem 3. Show that a general morphism

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f_B"] & B_2 \end{tikzcd}$$

of vector bundles is Cartesian if and only if it has the following universal property: Given a vector bundle $p\colon E \to B_1$ and a general morphism $g\colon E \to E_2$ covering f_B, there is a unique morphism $h\colon E \to E_1$ of vector bundles over B such that $g = f_E \circ h$:

$$\begin{tikzcd} E \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "p"'] & & \\ & E_1 \arrow[r, "f_E"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ & B_1 \arrow[r, "f_B"] & B_2 \end{tikzcd}$$

Problem 4. A *vector subbundle* of a vector bundle $p\colon E \to B$ is a subspace $E' \subset E$ which intersects each fibre of E in a vector subspace and which has the property that the restriction $p|\colon E' \to B$ is a vector bundle. What should one mean by the quotient vector bundle $E/E' \to B$? Give a definition and show that your definition does produce a vector bundle.

Problem 5. Let $\{U_\alpha\}$ be an open cover of a space B, and let

$$g_{\beta\alpha}\colon U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbf{R})$$

be continuous functions satisfying the cocycle condition

$$g_{\gamma\beta}(x) g_{\beta\alpha}(x) = g_{\gamma\alpha}(x) \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma$$

for all α, β and γ. Verify that the map $E(\{g_{\beta\alpha}\}) \to B$ defined in the lecture notes is an n-dimensional vector bundle. If the maps $g_{\beta\alpha}$ are the transition functions of a vector bundle $E \to B$, show that $E(\{g_{\beta\alpha}\})$ is isomorphic to E as a vector bundle over B.