Problem 1. For a space $X$, let $\text{Vect}_R^1(X)$ and $\text{Vect}_C^1(X)$ denote the sets of isomorphism classes of 1-dimensional real and complex vector bundles over $X$, respectively. Show that tensor product of vector bundles makes $\text{Vect}_R^1(X)$ and $\text{Vect}_C^1(X)$ into abelian groups. If $L$ is a line bundle over $X$, what are the transition functions of the inverse $L^{-1}$ in terms of those of $L$?

Problem 2. Let $\xi \to B$ be an $n$-dimensional real (complex) vector bundle equipped with a Riemannian (resp. Hermitian) metric. Show that every point in $B$ has a neighbourhood $U$ for which there exists a local trivialization of $\xi|_U$ compatible with the metric in the sense that for each $b \in U$, it takes the inner product on $\xi_b$ to the standard inner product on $\mathbb{R}^n$ (resp. the standard Hermitian inner product on $\mathbb{C}^n$). Conclude that $\xi$ has a system of transition functions taking values in in the orthogonal group $O(n) \subset GL_n(\mathbb{R})$ (resp. in the unitary group $U(n) \subset GL_n(\mathbb{C})$).

Problem 3. Check that the two constructions of the vector bundle $\text{Hom}(\xi, \zeta)$ discussed in the lectures produce isomorphic results.

Recall that continuous maps $f_0, f_1: X \to Y$ are homotopic if there exists a continuous map $h: X \times I \to Y$ such that $h(x, 0) = f_0(x)$ and $h(x, 1) = f_1(x)$ for all $x \in X$. Here $I$ denotes the unit interval $I = [0, 1]$. The following exercises sketch out a quick proof of the following theorem. (With more effort, we will later prove a more general version of the theorem.)

**Theorem.** Let $X$ be a compact Hausdorff space, and let $\xi$ be a vector bundle over a space $Y$. Suppose $f_0, f_1: X \to Y$ are homotopic maps. Then $f_0^*\xi$ and $f_1^*\xi$ are isomorphic.

First, recall the Tietze extension theorem: if $X$ is a normal space (such as a paracompact space) and $A \subset X$ is a closed subspace, then any continuous function $A \to \mathbb{R}$ can be extended to a continuous function $X \to \mathbb{R}$.

Problem 4. Suppose $X$ is paracompact, and let $\xi$ be a vector bundle over $X$. Suppose $A$ is a closed subspace of $X$, and let $s$ be a section of $\xi|A$. Using the Tietze extension theorem, show that $s$ can be extended to a section of $\xi$.

Problem 5. Let $\xi$ and $\zeta$ be vector bundles over a space $X$. Observe that vector bundle morphisms $\xi \to \zeta$ over $X$ can be identified with sections of $\text{Hom}(\xi, \zeta) \to X$. If $X$ is paracompact and $A \subset X$ is a closed subspace, use the previous problem to show that any isomorphism between $\xi|A$ and $\zeta|A$ can be extended to an isomorphism between $\xi|U$ and $\zeta|U$ for some neighbourhood $U$ of $A$.

Problem 6. Use the previous problem to show that if $X$ is a compact Hausdorff space and $\xi$ is a vector bundle over $X \times I$, then the vector bundles $\xi|X \times \{0\}$ and $\xi|X \times \{1\}$ are isomorphic. Use this to prove the theorem. **Hint:** compare $\xi$ and $\xi \times I$, where $\xi \to X$ is the restriction of $\xi$ to $X \times \{t\}$.