**INTRODUCTION TO TOPOLOGICAL $K$-THEORY**

**EXERCISE SESSION 6**

May 10, 2016

**Problem 1.** Show that the canonical line bundle $\gamma^1(\mathbb{R}^n)$ over $\mathbb{R}P^n$ is nontrivial for all $1 \leq n \leq \infty$. *Hint:* $\mathbb{R}P^1$ is homeomorphic to $S^1$.

**Problem 2.** Recall that a space $M$ is called an $n$-dimensional topological manifold if it is Hausdorff and every point in $M$ has a neighbourhood homeomorphic to $\mathbb{R}^n$. Prove that the Grassmannians $\text{Gr}_k(\mathbb{R}^n)$ and $\text{Gr}_k(\mathbb{C}^n)$ are topological manifolds of dimensions $k(n-k)$ and $2k(n-k)$, respectively. *Hint:* For $V \in \text{Gr}_k(\mathbb{R}^n)$, consider the neighbourhood $U_V \subset \text{Gr}_k(\mathbb{F}^n)$ consisting of those $W \in \text{Gr}_k(\mathbb{F}^n)$ which surject onto $V$ in the orthogonal projection of $\mathbb{F}^n$ onto $V$. Construct a homeomorphism between $U_V$ and the space of linear maps from $V$ to $V^\perp$.

Recall from the previous exercise session that for each $m$ and $n$, the map

$$
\perp : \text{Gr}_m(\mathbb{F}^{m+n}) \to \text{Gr}_n(\mathbb{F}^{m+n}), \quad V \mapsto V^\perp
$$

is a homeomorphism. Write $i$ for the inclusion maps

$$
i : \text{Gr}_k(\mathbb{F}^n) \hookrightarrow \text{Gr}_k(\mathbb{F}^{n+q})$$

induced by the inclusions $\mathbb{F}^n \hookrightarrow \mathbb{F}^{n+q}$ and $j$ for the maps

$$
j : \text{Gr}_k(\mathbb{F}^n) \longrightarrow \text{Gr}_{k+q}(\mathbb{F}^{n+q}),$$

sending $V \in \text{Gr}_k(\mathbb{F}^n)$ to the subspace $V \oplus \mathbb{F}^q \subset \mathbb{F}^n \oplus \mathbb{F}^q = \mathbb{F}^{n+q}$.

**Problem 3.** Define the (real) **bundle dimension** of a space $B$ to be the smallest integer $k \geq 0$ such that composition with the inclusion $\text{Gr}_m(\mathbb{R}^{m+n}) \hookrightarrow \text{Gr}_m(\mathbb{R}^\infty)$ induces a bijection $[B, \text{Gr}_m(\mathbb{R}^{m+n})] \to [B, \text{Gr}_m(\mathbb{R}^\infty)]$ for all $m, n \geq k$. If there is no such $k$, we say that the bundle dimension of $B$ is infinite. In what follows, let $B$ be a space with finite bundle dimension $k$.

(a) Show that any numerable vector bundle on $B$ admits a $k$-dimensional complement.

(b) Show that the inclusion $i : \text{Gr}_k(\mathbb{R}^{2k}) \hookrightarrow \text{Gr}_k(\mathbb{R}^{k+n})$ induces a bijection

$$[B, \text{Gr}_k(\mathbb{R}^{2k})] \overset{\approx}{\longrightarrow} [B, \text{Gr}_k(\mathbb{R}^{k+n})]$$

for all $n \geq k$.

(c) Show that the map $j : \text{Gr}_k(\mathbb{R}^{2k}) \to \text{Gr}_n(\mathbb{R}^{k+n})$ induces a bijection

$$[B, \text{Gr}_k(\mathbb{R}^{2k})] \overset{\approx}{\longrightarrow} [B, \text{Gr}_n(\mathbb{R}^{k+n})]$$

for all $n \geq k$. *Hint:* $\perp$.

(d) Show that any numerable vector bundle $\xi$ of dimension $n \geq k$ over $B$ splits as a direct sum $\xi \approx \eta \oplus \varepsilon^{n-k}$ for some $k$-dimensional numerable vector bundle $\eta$, and that such an $\eta$ is unique up to isomorphism.

(e) Show that if $\xi$ and $\zeta$ are two vector bundles of dimension $\geq k$ over $B$ such that $\xi \oplus \varepsilon^n$ and $\zeta \oplus \varepsilon^n$ are isomorphic for some $n \geq 0$, then $\xi$ and $\zeta$ are isomorphic.