INTRODUCTION TO TOPOLOGICAL $K$-THEORY
EXERCISE SESSION 7
May 24, 2016

Problem 1. Determine $\text{Vect}_R^n(S^1)$ for all $n \geq 1$.

Problem 2. Let

$$D^2_+ = \{(x_1, x_2, x_3) \in S^2 : x_3 \geq 0\}$$

and

$$D^2_- = \{(x_1, x_2, x_3) \in S^2 : x_3 \leq 0\}$$

be the upper and lower hemispheres of the sphere $S^2$, respectively, and observe that $D^2_+ \cap D^2_- = S^1$. For $x, y \in S^2$ with $y \neq -x$, write $R(x, y)$ for the rotation of $\mathbb{R}^3$ which sends $x$ to $y$ and fixes the subspace perpendicular to $x$ and $y$. Let $e_1, e_2$ and $e_3$ be the standard basis vectors for $\mathbb{R}^3$.

(a) Show that the tangent bundle $TS^2$ of $S^2$ can be obtained by the clutching construction applied to the trivial vector bundles $\varepsilon^2_\pm = D^2_\pm \times \mathbb{R}^2$ over $D^2_\pm$ and an isomorphism

$$\varphi: \varepsilon^2_+|S^1 \xrightarrow{\cong} \varepsilon^2_-|S^1$$

given by a continuous map $f: S^1 \rightarrow GL_2(\mathbb{R})$.

(b) Construct explicit trivializations $TS^2|D^2_\pm \xrightarrow{\cong} \varepsilon^2_\pm$ in terms of the functions $x \mapsto R(x, \pm e_3)$.

(c) Use your work from part (b) to find an explicit formula for the clutching map $f: S^1 \rightarrow GL_2(\mathbb{R})$.

Problem 3. Recall from Problem 3.1 that tensor product of vector bundles makes $\text{Vect}_C^1(X)$ into a group. Determine the group $\text{Vect}_C^1(S^2)$.

Problem 4. Determine $\text{Vect}_R^2(S^2)$.

The following problem gives an alternative construction of the Grothendieck group.

Problem 5. Let $M$ be a commutative monoid with addition $\oplus$. Consider the quotient $\text{Gr}'(M) = F/R$, where $F$ is the free abelian group generated by the set $M$ and $R$ is the normal subgroup of $F$ generated by all elements of the form $x \oplus y - x - y$ for $x, y \in M$. Show that $\text{Gr}'(M)$ is isomorphic to the Grothendieck group $\text{Gr}(M)$.