Problem 1. This problem outlines an alternative construction of the positive $K$-groups and the construction of external products from this point of view. We view $K^*(X)$ as $\mathbb{Z}$-graded. Recall from problem 10.5 that external product makes $\hat{K}^0(S^0) = \hat{K}^0(\text{pt})$ into a unital graded ring.

(a) Observe that for any $X$, the external product

$$\hat{K}^0(S^0) \otimes \hat{K}^0(X) \rightarrow \hat{K}^0(X)$$

makes $\hat{K}^0(X)$ into a graded $\hat{K}^0(S^0)$-module.

(b) Show that $\hat{K}^0(S^0) \approx \mathbb{Z}[b]$ as graded rings where $b$ has degree $-2$.

(c) Set

$$\hat{K}^*(X) = \mathbb{Z}[b^{\pm 1}] \otimes_{\mathbb{Z}[b]} \hat{K}^0(X).$$

Make sense of this object as a $\mathbb{Z}$-graded abelian group, and define the $n$-th $K$-group $\hat{K}^n(X)$ to be the degree $n$ part. Show that for $n \leq 0$, this agrees with the previous definition of $\hat{K}^n(X)$.

(d) Show that the external product

$$\hat{K}^0(X) \otimes \hat{K}^0(Y) \rightarrow \hat{K}^0(X \wedge Y)$$

factors as a composite

$$\hat{K}^0(X) \otimes \hat{K}^0(Y) \rightarrow \hat{K}^0(X) \otimes_{\mathbb{Z}[b]} \hat{K}^0(Y) \rightarrow \hat{K}^0(X \wedge Y).$$

(e) Let $R$ be a commutative ring, let $M$ and $N$ be $R$-modules, and let $S$ be a commutative ring equipped with a homomorphism $R \rightarrow S$. Then there is a natural isomorphism

$$S \otimes_R (M \otimes_R N) \approx (S \otimes_R M) \otimes_S (S \otimes_R N).$$

Use this and the previous part to construct an external product

$$\hat{K}^*(X) \otimes_{\mathbb{Z}[b^{\pm 1}]} \hat{K}^*(Y) \rightarrow \hat{K}^*(X \wedge Y).$$

Problem 2. Suppose $X$ is a path-connected compact Hausdorff space which can be written as the union of $n$ contractible closed subsets. Show that for all $x_1, \ldots, x_n \in \hat{K}(X)$, the product $x_1 \cdots x_n$ in $\hat{K}(X)$ is zero. Conclude that in particular the product on $\hat{K}(\Sigma Y)$ is trivial for any $Y$.

Problem 3. (The Cayley–Dickson construction of octonions) A $*$-algebra over the reals is a real vector space $A$ equipped with a bilinear multiplication $A \times A \rightarrow A$ and a linear map $*: A \rightarrow A$ satisfying $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. Show that for a $*$-algebra $A$, the direct sum $A \oplus A$ has a $*$-algebra structure given by

$$(a, b)(c, d) = (ac - d^*b, da + bc^*) \quad \text{and} \quad (a, b)^* = (a^*, -b).$$

Notice that iterating this construction starting with $\mathbb{R}$ (with $* = \text{id}$), one recovers the complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$. The octonions $\mathbb{O}$ are the result of applying the construction to the quaternions. Show that the octonions satisfy $a^*a = aa^* = |a|^2$ and $|ab| = |a||b|$, where $|\cdot|$ refers to the usual Euclidean norm on $\mathbb{R}^8$. Conclude that the octonions form a division algebra, and that multiplication by an octonion of unit length gives an orthogonal transformation of $\mathbb{R}^8$. Give an example to show that the octonions fail to be associative.