A primer on exterior powers

Let $F$ be a field. Write $\emptyset = \emptyset_F$

**Def 1:** Let $V$ be an $F$-vector space. The $k$-th exterior power of $V$ is the quotient

$$\Lambda^k V = \bigwedge^k V / a_k$$

where $T^k V = V^{\otimes k}$ ($T^0 V = F$) and $a_k \in V^{\otimes k}$

is the subspace generated by tensors $v_i \otimes \ldots \otimes v_i$, st.

$v_i = v_i$ for some $1 \leq i < k$.

**Ex 2:** $\Lambda^0 V = F$, $\Lambda^1 V = V$.

For a linear map $f : V \to W$, there is an evident induced map

$$f_* = \Lambda^k(f) : \Lambda^k V \to \Lambda^k W$$

$$[v_1 \otimes \ldots \otimes v_k] \mapsto [f(v_1) \otimes \ldots \otimes f(v_k)]$$

**Def 2:** Let $X, V$ be $F$-vector spaces. A $k$-linear map $f : V^k \to X$ is alternating if $f(v_1, \ldots, v_n) = 0$

whenever $v_i = v_i$ for some $1 \leq i < k$.

The following proposition is immediate from the definition of $\Lambda^k V$ and the analogous property for $T^k V$.

**Prop 3:** For any $F$-vector space $X$, composition with the alternating $k$-linear map
\[ V^k \longrightarrow \Lambda^k V \]

\[(v_1, \ldots, v_k) \longrightarrow [v_1 \wedge \ldots \wedge v_k] \]

gives a bijection

\[ \{ \text{linear maps } \Lambda^k V \rightarrow \mathbb{R} \} \cong \{ \text{alternating } k \text{-linear maps } V^k \rightarrow \mathbb{R} \} \]

Composition of tensors makes the direct sum

\[ T^*V = \bigoplus_{k=0}^{\infty} T^k V \]

into a graded ring. It is easy to check that the direct sum

\[ \alpha = \bigoplus_{k=0}^{\infty} \alpha_k \subset T^*V \]

is an ideal, so the direct sum/quotient

\[ \Lambda^* V = \bigoplus_{k=0}^{\infty} \Lambda^k V = \bigoplus_{k=0}^{\infty} T^k V / \alpha_k \]

\[ = \bigoplus_{k=0}^{\infty} \frac{T^k V}{\bigoplus_{k=0}^{\infty} \alpha_k} = T^*V / \alpha \]

has an induced graded ring structure. We write \( \wedge \) for the multiplication. For \( v_1, \ldots, v_k \in V \), the element

\[ v_1 \wedge \ldots \wedge v_k \in \Lambda^k V \]

is then simply the class of \( v_1 \wedge \ldots \wedge v_k \in T^k V \) in \( \Lambda^k V \). From the equation

\[ 0 = (v + w) \wedge (v + w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w \]

\[ = v \wedge w + w \wedge v \]

we conclude that

\[ (\wedge) \quad v \wedge w = - w \wedge v \]

for all \( v, w \in V \). It follows that \( \Lambda^* V \) is
graded commutative.

Suppose \( \{ e_i \} \) is a basis for \( V \) and pick a total order on \( I \). For a finite subset \( S \subset I \), we write

\[
es = e_{i_1} \wedge \ldots \wedge e_{i_k} \in \Lambda^k V
\]

where \( S = \{ i_1, \ldots, i_k \} \) and \( i_1 < \ldots < i_k \). (We interpret \( e_S = 1 \in \Lambda^0 V \).) Let

\[
P_k(I) = \{ S \subset I \mid |S| = k \}.
\]

Thus, for all \( k \geq 0 \), the set \( \{ e_S \} \subseteq P_k(I) \) is a basis for \( \Lambda^k V \).

**Proof:** Any element of \( \Lambda^k V \) can be written as a linear combination of products of the form

\[
v_1 \wedge \ldots \wedge v_k, \quad v_i \in V.
\]

Writing each \( v_i \) as a linear combination of the basis vectors \( e_i \) using distributivity, and using the relation (\(*\)) to reorder factors, we see that \( \Lambda^k V \) is spanned by the vectors \( e_S, \ S \in P_k(I) \).

It remains to show that the vectors

\[
es, \quad S \in P_k(I)
\]

are linearly independent. Suppose

\[
(\ast\ast) \quad \alpha_1 e_{S_1} + \ldots + \alpha_n e_{S_n} = 0
\]
where \( S_1, \ldots, S_n \) are distinct elements of \( P_n(I) \). Let \( J = \{ j_1, \ldots, j_d \} \) for some \( j_1 < \cdots < j_d \). Let \( P : V \to \mathbb{F}^d \) be the linear map sending \( e_{j_1} \) to the \( j_1 \)-th standard basis vector of \( \mathbb{F}^d \) and \( e_i \) to 0 for all \( i \in I \setminus J \).

Now the composite
\[
\bigwedge^d \mathbb{F}^d \to \mathbb{F}(\bigwedge^d \mathbb{F}^d) \to \mathbb{F}(\bigwedge^d (u_i, \ldots, u_d)) \to \det(u_i, \ldots, u_d)
\]
is an alternating \( d \)-linear map sending \( (e_{j_1}, \ldots, e_{j_d}) \) to 1, so we must have \( e_{j_i} \neq 0 \) in \( \bigwedge^d V \). For each \( S_i \), multiplying \((x_v)\) by \( e_{j_i} | S_i \) we obtain the equation
\[
\pm x_{S_i} e_{j_i} = 0,
\]
where \( x_{S_i} = 0 \). The claim follows. \( \Box \)

Cor 5: Spec. \( \dim V = n \). Then
\[
\dim \bigwedge^k V = \binom{n}{k} \text{ for } 0 \leq k \leq n
\]
and
\[
\bigwedge^k V = 0 \text{ for } k > n. \Box
\]

Thm 6: Let \( V, W \) be \( \mathbb{F} \)-vector spaces, and let \( i : V \hookrightarrow V \otimes W, j : W \hookrightarrow V \otimes W \) be the inclusions. Then the map
\[
\bigwedge^d (V) \otimes \bigwedge^d (W) \to \bigwedge^d (V \otimes W)
\]
is an isomorphism.
Prf: Pick bases \([e_a]_{a \in A}\) for \(V\) and \([e_b]_{b \in B}\) for \(W\). Then \([i_a(e_a)]_{a \in A}\) \([i_b(e_b)]_{b \in B}\) is a basis for \(V \oplus W\). Pick total orders on \(A\) and \(B\), and give \(A \sqcup B\) the induced total order where \(a < b\) for all \(a \in A, b \in B\). Using Thm 4, it is easy to verify that bases for the domain and the target derived from these bases correspond under the aforementioned maps. QED.