Lecture 2

Pullbacks, continued

Recall: A pullback of a v.b. \( p : E \to B \) along \( f : A \to B \) is a v.b. \( f^*E \to A \) together with a Cartesian morphism \( f^*E \to E \) covering \( f \) isos on fibres.

\[
\begin{array}{c}
\text{f}^*E \xrightarrow{f'} E \\
p' \downarrow \\
A \xrightarrow{f} B
\end{array}
\]

(Often the morphism is left implicit.)

Intuition: \( f^*E \) is the v.b. over \( A \) whose fibre over \( a \in A \) "is" the fibre of \( E \) over \( f(a) \). Here "is" should not be taken literally; it is given content by the map \( f' \).

E.g. If \( E \xrightarrow{p} B \) v.b., \( A \subset B \) subspace, then \( E|A = p^*(A) \xrightarrow{p'} A \) is a pullback of \( E \to B \) along \( A \subset B \).
Next goal: Pullbacks along arbitrary maps exist and are unique up to unique isomorphism.

Suppose \( B \xrightarrow{p} A \) is a v.b., \( f: A \rightarrow B \) a map.

**Existence of** \( f^* E \)\( f^* E = \{(a, v) \in A \times E \mid p(v) = f(a)\} \subset A \times E \)

(\textit{fibre product of } A \textit{ and } E \textit{ over } B \).

Then

\[
\begin{array}{ccc}
A \times_B E & \xrightarrow{f'} & E \\
\downarrow{p'} & & \downarrow{p} \\
A & \xrightarrow{f} & B
\end{array}
\]

commutes. We will show that this diagram gives the desired pullback.

Let \( \Gamma_f := \{(a, f(a)) : a \in A \times B \} \subset A \times B \) (graph of \( f \)).
Thus $p'$ factors as

vector bundle, restricting of $A \times E \to A \times B$ to $\Gamma_f$

\[
\begin{array}{cccccc}
A \times E & \xrightarrow{\nu} & \Gamma_f & \xrightarrow{\alpha} & A \\
(a, v) & \mapsto & (a, p(v)) = (a, f(a)) & \mapsto & a
\end{array}
\]

So $p' : A \times B \times E \to A$ is a u.b. Moreover,

for any $a_0 \in A$, the map

\[
\begin{array}{ccc}
f' |_{a_0} & (A \times B \times E)_{a_0} & \longrightarrow & E_{f(a_0)} \\
(\nu_{a_0, v}) & \mapsto & \{(u, v) \mid u \in E, p(u) = f(a_0)\}
\end{array}
\]

is clearly an iso.

Conclusion: can take $f^*E = A \times B \times E$.

**Uniqueness of $f^*E$:**

Suppose

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f_1} & E & \xleftarrow{f_2} & E_2 \\
\beta_1 & \xrightarrow{f} & \beta_2
\end{array}
\]

are two pullbacks of $p:E \to B$ along $f$.

**Claim:** $f$ unique iso $g:E_1 \to E_2$ of u.b.'s over $A$ s.t. $f_1 = f_2 \circ g$. 
If \( g \) exists, it is necessarily given by
\[
(E_1)_a \xrightarrow{f_1 a} E_{f(a)} \xrightarrow{(f_{E a})^{-1}} (E_2)_a \quad (a \in A)
\]
on fibres. It remains to show that the map \( g \) so defined is continuous. This follows easily by passing to local trivializations.

(Do it!) \( \Box \)

The following lemma relates Cartesian morphisms (as defined by us) to how they might be defined in more general contexts.

**Lemma 1.** A general morphism

\[
\begin{array}{c}
E_1 \xrightarrow{f_E} E_2 \\
\beta_1 \xrightarrow{f_{\beta}} \beta_2
\end{array}
\]

is Cartesian iff it has the following universal property: Given a v.b. \( E \xrightarrow{p} \beta_1 \) and a general morphism \( g : E \to E_2 \) covering \( f_{\beta} \), there exists a unique morphism \( h : E \to E_1 \) of v.b.'s over \( \beta_1 \) s.t. \( g = f_E \circ h \).

\[
\begin{array}{c}
E \xrightarrow{g} E_1 \xrightarrow{f_E} E_2 \\
\beta \xrightarrow{p} \beta_1 \xrightarrow{f_{\beta}} \beta_2
\end{array}
\]

**Pf:** Exercise. \( \Box \)
Transition functions

Suppose $E \xrightarrow{p} B$ is a u.b. and let $\{U_\alpha\}$ be an open cover of $B$ with local trivializations

$$h_\alpha : p^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$$

The composite

$$((U_\alpha \cap U_\beta) \times \mathbb{R}^n) \xrightarrow{h_\alpha^{-1} \circ (U_\alpha \cap U_\beta)} (U_\alpha \cap U_\beta) \times \mathbb{R}^n \xrightarrow{h_\beta^{-1}} (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is of the form

$$(x, v) \mapsto (x, g_{\beta\alpha}(x)v)$$

for some uniquely determined continuous

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \text{GL}_n(\mathbb{R})$$

The maps $g_{\beta\alpha}$ are called transition functions.

They satisfy the cocycle condition

$$g_{\beta\alpha}(x)g_{\alpha\gamma}(x) = g_{\alpha\beta}(x) \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

We can recover $E$ from the $g_{\beta\alpha}$'s:
Let $E(\{g_{\alpha\beta}\}) = \bigcup_{\alpha} U_{\alpha} \times \mathbb{R}^n/n$

where $n$ is defined by setting

$((x, u) \in U_{\alpha} \times \mathbb{R}^n) \sim ((x, g_{\alpha\beta}(x)u) \in U_{\beta} \times \mathbb{R}^n)$

for all $x \in U_{\alpha} \cap U_{\beta}, u \in \mathbb{R}^n$.

We have the map

$E(\{g_{\alpha\beta}\}) \rightarrow B$

$[((x, u) \in U_{\alpha} \times \mathbb{R}^n)] \rightarrow x$

Prop 2: $E(\{g_{\alpha\beta}\}) \rightarrow B$ is a v.b. isomorphic to $E \rightarrow B$.

Pf: Exercise.

Conclusion: $E$ can be obtained by gluing together trivial vector bundles in the way described by the transition functions.

We can turn this idea into a method for constructing v.b.'s.

Prop 3: Given an open cover $\{U_{\alpha}\}$ of $B$ and continuous functions

$g_{\alpha\beta} : U_{\alpha} \cap U_{\beta} \rightarrow \text{GL}_n(\mathbb{R})$

satisfying the cocycle condition, the map $E(\{g_{\alpha\beta}\}) \rightarrow B$ is an $n$-dim v.b.

Pf: Exercise.
Constructions on vector bundles

Direct product If $\xi \xrightarrow{p_\xi} B$ and $\zeta \xrightarrow{p_\zeta} C$ are v.b.'s, the direct product of $\xi$ and $\zeta$ is the v.b.

$$\xi \times \zeta \xrightarrow{p_\xi \times p_\zeta} B \times C.$$

Fibres $$(\xi \times \zeta)(b, c) = \xi_b \times \zeta_c$$

can be local trivializations by taking products of local trivializations for $\xi$ and $\zeta$.

Direct sum $\xi \xrightarrow{p_\xi} B$, $\zeta \xrightarrow{p_\zeta} B$ v.b.'s.

The direct sum or Whitney sum of $\xi$ and $\zeta$ is

$$\xi \oplus \zeta = \{(v, w) \in \xi \times \zeta \mid p_\xi(v) = p_\zeta(w) \} \xrightarrow{\{(v, w) \mid p_\xi(v) = p_\zeta(w) \}} B$$

This is a v.b., since it agrees with

$$(\xi \times \zeta) \times_{B \times B} B = \Delta^*(\xi \times \zeta), \quad \Delta : B \rightarrow B \times B$$

diagonal.

$$\xi \oplus \zeta \xrightarrow{\Delta} \xi \times \zeta$$

$$\downarrow \Delta \downarrow \Delta$$

$$\xi \oplus \zeta \xrightarrow{\Delta} B \times B$$

Fibres $$(\xi \oplus \zeta)_b \cong \xi_b \oplus \zeta_b.$$
Generalizing $\Theta$, we have

**Meta-theorem 4**

i) All the usual constructions on vector spaces

$(\oplus, \otimes, \text{Hom}(-,-), \Delta^k, (-)^*, \ldots)$

generalize to vector bundles by performing them fibre-wise.

ii) Natural isomorphisms b/w these constructions

(e.g. $V_1 \otimes (V_2 \otimes V_3) \cong (V_1 \otimes V_2) \otimes V_3$, $V^{**} \cong V$, ...)

generalize to natural isomorphisms b/w the corresponding constructions on v.b.'s

iii) The constructions on v.b.'s commute

with taking pullbacks (so e.g.

$\text{Hom}(f^*g, f^*h) \cong (f^*) \text{Hom}(g, h)$)

Instead of trying to formalize this and
giving a complete proof, let us just
discuss an example to illustrate the result.

Suppose $\xi \to B$, $\zeta \to B$ are v.b.'s of dimension, respectively. We will construct the v.b.

$\text{Hom}(\xi, \zeta) \to B$ in two different ways.
First, a bit of notation: if $f : V_1 \to W_1$ and $g : W_1 \to W_2$ are linear maps, write $\text{Hom}(f,g)$ for the linear map

$$\text{Hom}(f,g) : \text{Hom}(V_1, W_1) \to \text{Hom}(V_2, W_2)$$

A $A \mapsto g \circ A \circ f$

Construction 1

Choose an open cover \{U_\alpha\} of $\mathcal{B}$ and local trivializations of $\mathcal{G}$ and $\mathcal{G}$ over each $U_\alpha$. Let

$$g_{\alpha \beta} : U_\alpha \cap U_\beta \to \text{Gl}_n(\mathbb{R})$$

$$g_{\beta \alpha} : U_\beta \cap U_\alpha \to \text{Gl}_n(\mathbb{R})$$

Let the resulting transition functions.

Now the functions (\text{linear automorphisms of $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$})

$$g_{\alpha \beta} : U_\alpha \cap U_\beta \to \text{Gl}(\text{Hom}(\mathbb{R}^n, \mathbb{R}^n))$$

$$x \mapsto \text{Hom}(g_{\beta \alpha}^{-1}(x), g_{\alpha \beta}(x))$$

are continuous and satisfy the cocycle condition (check!), so they define a v.i.b. $\mathcal{B}$. This is $\text{Hom}(\mathcal{G}, \mathcal{G})$. 

Re: The essential properties of $\text{Hom}(-,-)$

("the construction on vector spaces") needed in this construction are

1) Continuity: $\text{Hom}(f, g)$ depends continuously on $f$ and $g$.

2) Functionality: $\text{Hom}(\text{id}_V, \text{id}_W) = \text{id}_{\text{Hom}(V, W)}$
and

$$\text{Hom}(f_2, g_2) \circ \text{Hom}(f_1, g_1) = \text{Hom}(f_1 \circ f_2, g_2 \circ g_1)$$

for composable linear maps $f_1, f_2$ and $g_1, g_2$.

(Among other things, this implies that $\text{Hom}(f, g)$ is an isomorphism when $f$ and $g$ are.)

Next: the second construction.