Lecture 3

Constructions on vector bundles, continued

\( g \to B, \sigma \to B \) u.b. \( \sigma \) of dim. \( n, m \), respectively.

Last time: a construction of \( \text{Hom}(g, \sigma) \to B \)
in terms of transition functions.

Let us discuss another way to construct this u.b.

**Construction 2**

We want \( \text{Hom}(g, \sigma)_b = \text{Hom}(g_b, \sigma_b) \), so let us define \( \text{Hom}(g, \sigma) \) to be

\[
\text{Hom}(g, \sigma) = \bigsqcup_{b \in B} \text{Hom}(g_b, \sigma_b)
\]
as a set, and let \( p \) to be the projection

\[
p : \text{Hom}(g, \sigma) \longrightarrow B
\]

which sends \( \text{Hom}(g_b, \sigma_b) \) to \( b \in B \). Then

\[
p^{-1}(b) = \text{Hom}(g_b, \sigma_b)
\]

and what remains is to give \( \text{Hom}(g, \sigma) \) a topology making \( p \) into a vector bundle.

Given an open \( U \subset B \) and local trivializations

\[
h^g : g \mid_U \cong U \times \mathbb{R}^n
\]

\[
h^s : \sigma \mid_U \cong U \times \mathbb{R}^m
\]
we get a bijection
\[ H(h^8, h^5) : \text{Hom}(\mathbb{S}^8, \mathbb{S}^5) \mid U \rightarrow U \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \]
\[ A \rightarrow (\rho(A), \text{Hom}(h^8_{\rho(A)}, h^5_{\rho(A)})(A)) \]

where
\[ h^8_b : \mathbb{S}^8 \rightarrow \mathbb{R}^n, \quad b \in U \]
\[ h^5_b : \mathbb{S}^5 \rightarrow \mathbb{R}^m, \quad b \in U \]
are the maps defined by \( h^8 \) and \( h^5 \).

Define a topology on \( \text{Hom}(\mathbb{S}^8, \mathbb{S}^5) \mid U \) by requiring that \( H(h^8, h^5) \) is a homomorphism.

**Check:**
- \( T_\nu \) is independent of the choice of local trivializations \( h^8, h^5 \) over \( U \)
  (different choices as maps \( H(h^8, h^5) \) related by a homomorphism of \( U \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \))
- If \( \nu \subset U \) is open, \( T_\nu \) agrees with the subspace topology inherited from \( T_\nu \) (can restrict trivializations from \( U \) to \( \nu \))

**Conclude:** The topologies \( T_\nu \) for varying \( U \) patch together to give a topology on \( \text{Hom}(\mathbb{S}^8, \mathbb{S}^5) \) for which \( T_\nu \) agrees with the subspace topology on \( \text{Hom}(\mathbb{S}^8, \mathbb{S}^5) \mid U \). Thus \( H(h^8, h^5) \)'s give local trivializations, and \( \text{Hom}(\mathbb{S}^8, \mathbb{S}^5) \rightarrow B \) is a v.b.
Again, the essential features of \( \text{Hom}(\mathbf{-}, \cdot ) \) (the construction on vector spaces) making this work are the continuity and functionality of \( (f, g) \mapsto \text{Hom}(f, g) \).

For the second construction, have

\[
\text{Hom}(8, 8)_\beta = \text{Hom}(8 \otimes 5_\beta).
\]

This makes parts (ii) & (iii) of Metathesis 4 Lemma 2 transparent. For example, for (iii), we can define a Cartesian \( \bar{f} \)

\[
\text{Hom}(f^*, f^*) \xrightarrow{\bar{f}} \text{Hom}(8, 8)
\]

by taking it to be the composite

\[
\text{Hom}(f^*, f^*) = \text{Hom}(f^*a, f^*a) \xrightarrow{\pi} \text{Hom}(g_\beta a, g_\beta a) = \text{Hom}(8, 8)_{\beta(a)}
\]

(induced by \( f^*a \xrightarrow{\pi} f^*a \)

\[
g^*a \xrightarrow{\pi} g^*a
\]

(\(a \in A\)) on fibres.

**Exercise:** Check that the two constructions of \( \text{Hom}(8, 8) \) agree.

**Exercise:** Check that when applied to \( 0 \), the analogous constructions recover the Whitney sum of \( v \otimes v \).
Kernel, image and cokernel

Recall: If $f: V \rightarrow W$ is a linear map,

$\text{Ker} f = \{ v \in V \mid f(v) = 0 \} \subset V$

$\text{Im} f = \{ f(v) \mid v \in V \} \subset W$

$\text{Coker} f = W / \text{Im} f$

We would like to generalize these notions to vector bundles. Suppose $f: \mathcal{E} \rightarrow \mathcal{F}$ is a map of vector bundles over $B$. Define

$\text{Ker} f = \{ v \in \mathcal{E} \mid f(v) = 0 \} \subset \mathcal{E}$

$\text{Im} f = \{ f(v) \mid v \in \mathcal{E} \} \subset \mathcal{F}$

$\text{Coker} f = \mathcal{E} / \text{Ker} f$ where $w_1 = w_2$ if $w_1 - w_2 \in \text{Im} f$ belong to the same fibre and $w_1 - w_2 \in \text{Ker} f$.

We have evident projections from these spaces to $B$, and the fibres over $b \in B$ are $\text{Ker}(f|_b)$, $\text{Im}(f|_b)$ and $\text{Coker}(f|_b)$, respectively.

Problem: In general, $\mathcal{F}$ has local trivializations.
The dimensions of \( \ker(f|_B) \), \( \text{Im}(f|_B) \) and \( \text{Clem}(f|_B) \) jump at \( t = 0 \), so \( f \) is \textit{not} local trivialization near \( t = 0 \).

**Def** A map of v.b.'s \( /B \) is of \textit{constant rank} \( k \) if \( f|_B : S_0 \to S_0 \) is of rank \( k \) for all \( B \in B \).

(Recall: \textit{The rank} of a linear map \( f : V \to W \) is \( \dim(\text{Im} f) = \dim(V) - \dim(\ker f) = \dim(W) - \dim(\text{Clem} f) \).

**Prop** Suppose \( f : S \to S \) is of constant rank \( k \). Then \( \ker f \to B \), \( \text{Im} f \to B \) and \( \text{Clem} f \to B \) are vector bundles.

**Pf:** It is enough to show local triviality.

The problem is local; so we may assume \( S = B \times \mathbb{R}^n \), \( S = B \times \mathbb{R}^m \).

Then \( f : S \to S \) has the form

\[
\begin{array}{c}
0 \times \mathbb{R}^n & \xrightarrow{f} & 0 \times \mathbb{R}^m \\
(l, v) & \mapsto & (l, f_0(v))
\end{array}
\]

for some continuous \( B \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m) \).
Fix $b_0 \in B$ and decompose
\[ R^n = V_1 \oplus V_2, \quad V_2 = \text{Ker} f_{b_0} \]
\[ R^m = W_1 \oplus W_2, \quad W_1 = \text{Im} f_{b_0}. \]

Let
\[ A_b = \begin{pmatrix} f_b & 0 \\ 0 & 1 \end{pmatrix} : V_1 \oplus V_2 \oplus W_2 \rightarrow W_1 \oplus W_2 \oplus V_2. \]

Then the map $B \rightarrow \text{Ker} (V_1 \oplus V_2 \oplus W_2, W_1 \oplus W_2 \oplus V_2)$ $b \mapsto A_b$ is continuous and $A_{b_0}$ is an isomorphism, so $A_b$ is an isomorphism for all $b$ in a neighbourhood $U$ of $b_0$.

Let us construct local trivializations over $U$:

\[ \text{Ker} f : \begin{cases} A_b(v_1, v_2, 0) = v_2 \text{ for } (v_1, v_2) \in \text{Ker} f_b \\ A_b \text{ invertible} \end{cases} \]

\[ \Rightarrow A_b : \text{Ker} f_0 \rightarrow V_2 \text{ mono} \]

\[ \Rightarrow \quad \text{dim comparison} \quad U \quad \text{iso} \]

\[ \Rightarrow \text{dim} f_1 V \quad \approx \quad U \times V_2 \]
\[ (b, v_1, v_2) \text{ for } (b, A_b(v_1, v_2, 0)) \]
\[ (b, A_b^{-1}(0, v_2)) \text{ for } (b, v_2) \]

local triv.
\[ \text{Im } f : \begin{cases} A_{b}(v_1, 0, 0) = f_{b}(v_1) \text{ for } v_1 \in V_1 \\ A_{b} \text{ invertible} \end{cases} \]

\[ \Rightarrow A_{b} : V_1 \longrightarrow \text{Im } (f_{b}) \text{ mono} \]

\[ \text{dim comparison } \hline \text{iso} \]

\[ \text{colim } f | \Upsilon \xrightarrow{\cong} \Upsilon \times V_1 \]

\[ (b, A_{b}(v_1, 0, 0)) \longmapsto (b, v_1) \]

\[ (b, w_1, w_2) \longmapsto (b, A_{b}^{-1}(w_1, w_2, 0)) \]

\( \text{local triv.} \)

\[ \text{colim } f : A_{b} \text{ restricts to a mono} \]

\[ \tilde{A}_{b} = A_{b} : V_1 \oplus W_2 \longrightarrow W_1 \oplus W_2 \]

\[ \text{Dimension comparison } \Rightarrow \text{ this is an iso. } \]

\[ \text{Discussion for } \text{Im } f \Rightarrow \text{ under this iso, } V_1 \cong \text{Im } f_{b} \]

\[ \Upsilon \times (V_1 \oplus W_2 / V_1) \xrightarrow{\cong} \text{colim } f | \Upsilon \]

\[ (b, (v_1, w_2) + V_1) \longmapsto (\tilde{A}_{b}(v_1, w_2) + \text{Im } f_{b}) \in \left( \text{colim } f \right)_{b} \]

\[ (b, \tilde{A}_{b}^{-1}(w_1, w_2) + V_1) \longmapsto ((w_1, w_2) + \text{Im } f_{b}) \in \left( \text{colim } f \right)_{b} \]

\( \text{local triv.} \)
Recall: A sequence
\[ M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \]
of groups / modules / vector spaces is exact if \( \text{Ker} f_2 = \text{Im} f_1 \). A sequence
\[ M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n \]
is exact if it is exact at every \( M_i \), \( 2 \leq i \leq n-1 \).
A short exact sequence is an exact sequence of the form
\[ 0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0 \]

We generalize the language to vector bundles and constant-rank morphisms. In the context of vector bundles \( /B \), \( 0 \) should be understood as the \( 0 \)-dim\'l trivial v.b. \( /B \), i.e., \( B \xrightarrow{\text{id}} B \).

Exercise: If \( 0 \rightarrow \xi \xrightarrow{i} \xi \xrightarrow{p} \eta \rightarrow 0 \) is a short exact sequence of vector bundles \( /B \), then \( \xi \cong \text{Ker}(p) \) and \( \xi \cong \text{Cok}(i) \).