Homotopy invariance of pullbacks of vector bundles (cont.)

Recall: we are trying to show

Thm 1: Sp: $\mathcal{F} \to \mathcal{X}$ is a numerable v.b. Let $f, g: \mathcal{X} \to Y$ be homotopic maps. Then the v.b.'s $f^*\mathcal{F}$ and $g^*\mathcal{F}$ over $\mathcal{X}$ are isomorphic.
We have reduced the thm to showing

Lemma 2: Let $\mathcal{F} \to \mathcal{X} \times I$ be a numerable v.b.
Then $\exists$ Cartesian morphism $f: \mathcal{F} \to \mathcal{F}$ covering
the map $r: \mathcal{X} \times I \to \mathcal{X} \times I$, $(x, t) \mapsto (x, 1)$.

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{r} & \mathcal{F} \\
\downarrow & & \downarrow \\
\mathcal{X} \times I & \xrightarrow{r} & \mathcal{X} \times I
\end{array}
$$

Last time, we proved

Lemma 3: Let $\mathcal{F} \to \mathcal{X} \times I$ be a numerable v.b.
Then $\exists$ numerable cover $\{U_p\}_{p \in \mathcal{B}}$ of $\mathcal{X}$ s.t.
$\mathcal{F}|_{U_p \times I}$ is trivial $\forall p \in \mathcal{B}$.

Pf of Lemma 2: Let $\{U_p\}$ be a cover of $\mathcal{X}$
of the type afforded by Lemma 3, and
choose local trivializations $\nu_p: \mathcal{F}|_{U_p \times I} \to \mathcal{V}_p \times \mathcal{R}^n$
for $p \in \mathcal{B}$ and a p.o.v. $\{\nu_p\}_{p \in \mathcal{B}}$ subordinate to $\{U_p\}_{p \in \mathcal{B}}$. 
For $\beta \in \mathcal{B}$, define a Cartesian morphism

\[ \begin{array}{ccc}
\beta & \xrightarrow{f_\beta} & \beta \\
\beta & \xrightarrow{id_B} & \beta \\
\beta \times I & \xrightarrow{r_\beta} & \beta \times I
\end{array} \]

as follows:

* $r_\beta$ is the map $r_\beta(x, t) = (x, \min(1, t + \psi_\beta(x)))$.
* Over $\beta \times I$, $f_\beta$ is the map

\[ \beta \times I \rightarrow \beta \times I \]

which, under $h_\beta$, corresponds to the map

\[ \beta \times I \times \mathbb{R}^n \rightarrow \beta \times I \times \mathbb{R}^n \\
(x, t, \mathbf{v}) \mapsto (r_\beta(x, t), \mathbf{v}) \]

* Outside $\text{supp}(\psi_\beta)$, $f_\beta$ is the identity.

Check: this gives a well-defined Cartesian morphism covering $r_\beta$.

Pick a total order on $\mathcal{B}$. Using this order, we can make sense of the composite

\[ \beta \in \mathcal{B} \ni f_\beta : \beta \rightarrow \beta \]

as follows: For each $x \in \mathcal{I}$, we can find a neighborhood $V \in \mathcal{U}$ of $x \wedge i$. $V$ meets $\text{supp}(\psi_\beta)$ for only finitely many $\beta \in \mathcal{B}$, say for $\beta_1, \ldots, \beta_k \in \mathcal{B}$, where $\beta_1 < \beta_2 < \cdots < \beta_k$. 
We let 
\[(o_{f^k})(v) = (f^1, \ldots, f^n)(v) \text{ for } v \in V \times I.\]

Check: This gives a well-defined map \(o_{f^k}: E \to \mathfrak{g}.\)

Now \(f = \bigcirc_{o f^k}\) is as desired. \(\square\)

This concludes the proof of Thm 1.

Next goal:

Thm 4: There is a bijection

\[\text{Vec}^k_{\mathbb{F}}(X) \cong [X, \text{Gr}_k(\mathbb{F}^n)]\]

\[\begin{array}{c}
\text{Vec}^k_{\mathbb{F}}(X) \\
\text{[X, Gr}_k(\mathbb{F}^n)] \\
\end{array}\]

Here \(\mathbb{F} = \mathbb{R}\) or \(\mathbb{C}\), and the space \(\text{Gr}_k(\mathbb{F}^n)\)

( the Grassmannian of \(k\)-planes in \(\mathbb{F}^n\)) and

the vector bundle \(\gamma^k \to \text{Gr}_k(\mathbb{F}^n)\) are defined below. Thm 1 \(\Rightarrow\) the above map is well-defined.

Rmk: Everything we have done so far in this course has worked equally well for both real and complex v.b.'s, even though in the notation we have occasionally assumed the real case. From now on, let us adopt the convention that \(\mathbb{F}\) stands for either \(\mathbb{R}\) or \(\mathbb{C}\).
Let
\[ \mathbb{F}^\infty = \{ (a_1, a_2, \ldots) \mid a_i \in \mathbb{F}, a_i \neq 0 \text{ for only finitely many } i \} . \]
This is an \( \mathbb{F} \)-vector space. For \( 0 \leq n \leq \infty \), \( \mathbb{F}^n \) has the inner product
\[ \langle (a_i)_{i=1}^n, (b_i)_{i=1}^n \rangle = \sum_{i=1}^n a_i b_i ; \]
and \( \mathbb{C}^n \) the Hermitian inner product
\[ \langle (a_i)_{i=1}^n, (b_i)_{i=1}^n \rangle = \sum_{i=1}^n a_i \overline{b_i} . \]
For \( 0 \leq n \leq \infty \), define
\[ \mathrm{Gr}_n(\mathbb{F}^n) = \{ \text{\( k \)-dim'l vector subspaces of } \mathbb{F}^n \} , \]
the Grassmannian of \( k \)-planes in \( \mathbb{F}^n \), and
\[ g^k = g^k(\mathbb{F}^n) = \{ (V, v) \in \mathrm{Gr}_n(\mathbb{F}^n) \times \mathbb{F}^n \mid v \in V \} , \]
\[ g^k(\mathbb{F}^n) \xrightarrow{p} \mathrm{Gr}_n(\mathbb{F}^n) \]
\[ (V, v) \xrightarrow{p} V \]
(so \( p^{-1}(V) = \{ V \} \times V \), "the vector space \( V \) itself")

Goal: Give \( \mathrm{Gr}_n(\mathbb{F}^n) \) and \( g^k(\mathbb{F}^n) \) topologies
making \( p: g^k(\mathbb{F}^n) \to \mathrm{Gr}_n(\mathbb{F}^n) \) into a numerable \( \mathbb{F} \)-v.b., the \underline{canonical} / \underline{universal} v.b. over
\( \mathrm{Gr}_n(\mathbb{F}^n) \).
Let

$$V_k(\mathbb{F}^n) = \{ (v_1, \ldots, v_k) \in (\mathbb{F}^n)^k \mid v_1, \ldots, v_k \text{ linearly independent} \}.$$  

(Such $V_k(\mathbb{F}^n)$'s are called \textit{Stiefel manifolds}.) Then $V_k(\mathbb{F}^n) \subset (\mathbb{F}^n)^k$ is an open subspace.

We have a surjection

$$q : V_k(\mathbb{F}^n) \longrightarrow Gru_k(\mathbb{F}^n)$$

$$(v_1, \ldots, v_k) \longrightarrow \text{span}(v_1, \ldots, v_k)$$

and we give $Gru_k(\mathbb{F}^n)$ the quotient topology from $V_k(\mathbb{F}^n)$. Inside $V_k(\mathbb{F}^n)$, we have the subspace

$$V_k^0(\mathbb{F}^n) = \{ (v_1, \ldots, v_k) \in V_k(\mathbb{F}^n) \mid \langle v_i, v_j \rangle = \delta_{ij} \}.$$  

(orthonormal $k$-frames)

The restriction $q|_k : V_k^0(\mathbb{F}^n) \rightarrow Gru_k(\mathbb{F}^n)$ is also a surjection, and diagram

$$\begin{array}{ccc}
V_k^0(\mathbb{F}^n) & \longrightarrow & V_k(\mathbb{F}^n) \\
\downarrow q|_k & & \downarrow q \\
Gru_k(\mathbb{F}^n) & \longrightarrow & Gru_k(\mathbb{F}^n)
\end{array}$$

The quotient topologies on $Gru_k(\mathbb{F}^n)$ induced by $q$ and $q|_k V_k^0(\mathbb{F}^n)$ agree.
$V_u^o(\mathbb{F}^n)$ is a closed subset of $S(\mathbb{F}^n)^\times$ where $S(\mathbb{F}^n)$ denotes the unit sphere in $\mathbb{F}^n$. Thus $V_u^o(\mathbb{F}^n)$ and hence $\text{Gr}_u(\mathbb{F}^n) = \mathbb{F} V_u^o(\mathbb{F}^n)$ are compact. The map

$$\text{Gr}_u(\mathbb{F}^n) \xrightarrow{g} \text{End}(\mathbb{F}^n) \cong \mathbb{F}^{n^2}$$

is injective, and it is continuous since the composite

$$V_u^o(\mathbb{F}^n) \xrightarrow{g^{-1}} \text{Gr}_u(\mathbb{F}^n) \xrightarrow{g} \text{End}(\mathbb{F}^n)$$

is monotone. Since $\text{Gr}_u(\mathbb{F}^n)$ is compact and $\text{End}(\mathbb{F}^n)$ is Hausdorff, the map $g$ is an embedding. In particular, $\text{Gr}_u(\mathbb{F}^n)$ is Hausdorff.

Give $\mathfrak{s}^e(\mathbb{F}^n) \subset \text{Gr}_u(\mathbb{F}^n) \times \mathbb{F}^n$ the subspace topology. Then the map $p : \mathfrak{s}^e(\mathbb{F}^n) \to \text{Gr}_u(\mathbb{F}^n)$ is continuous, and we have already seen that the fibres of $p$ are vector spaces.

To see that $p : \mathfrak{s}^e(\mathbb{F}^n) \to \text{Gr}_u(\mathbb{F}^n)$ is a V.I., it remains to check local triviality. Given $V \in \text{Gr}_u(\mathbb{F}^n)$, write $g_V : \mathbb{F}^n \to V$ for the orthogonal projection. Then

$$V_V = \{ W \in \text{Gr}_u(\mathbb{F}^n) \mid g_V(W) = V \}$$
is an open neighbourhood of \( V \), and
\[
y^*(\mathbb{F}^n)|U_V \rightarrow U_V \times V
\]
\[
(W, w) \mapsto (W, y_V(w))
\]
is a local trivialization. Thus \( y^*(\mathbb{F}^n) \rightarrow \text{Gr}_n(\mathbb{F}^n) \)
is a v.b. Since \( \text{Gr}_n(\mathbb{F}^n) \) is compact Hausdorff, \( y^*(\mathbb{F}^n) \) is automatically numerable.

Case \( n = \infty \):

Def 5: Let \( \mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots \) be an ascending sequence of spaces (each with the subspace topology from the next), and let \( \mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n \).

The weak or direct limit topology on \( \mathcal{X} \) is the topology in which \( U \subset \mathcal{X} \) is open if and only if \( U \cap \mathcal{X}_n \) is open in \( \mathcal{X}_n \).

(Equivalently, \( C \subset \mathcal{X} \) is closed in \( \mathcal{X} \) if and only if \( C \cap \mathcal{X}_n \) is closed in \( \mathcal{X}_n \).

Check: This is a topology?

Exc: The subspace topology \( \mathcal{X}_n \subset \mathcal{X} \) inverts from the direct limit topology on \( \mathcal{X} \) agrees with the original topology on \( \mathcal{X}_n \).

Exc: For any space \( \mathcal{Y} \), a function \( f: \mathcal{X} \rightarrow \mathcal{Y} \) is continuous w.r.t. the direct limit topology if and only if \( f_n: \mathcal{X}_n \rightarrow \mathcal{Y} \) is continuous for all \( n \).
The inclusions \( F^n \subset F^{n+1} \) \((a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, 0)\) induce a sequence of inclusions
\[
\text{Cir}_n(F^n) \subset \text{Cir}_n(F^{n+1}) \subset \cdots
\]
and we have \( \bigcup_n \text{Cir}_n(F^n) = \text{Cir}_n(F^\infty) \).
Give \( \text{Cir}_n(F^\infty) \) the direct limit topology. Give \( F^\infty \) the direct limit topology from
\[
F^1 \subset F^2 \subset \cdots
\]
the space \( \text{Cir}_n(F^\infty) \times F^\infty \) the product topology, and \( \gamma_n(F^\infty) \subset \text{Cir}_n(F^\infty) \times F^\infty \) the subspace topology. Then \( p : \gamma_n(F^\infty) \to \text{Cir}_n(F^\infty) \) is continuous.

Next time: Local triviality of \( \gamma_n(F^\infty) \).