Pairs and pointed spaces (cont.)

Def 1: If $X$ is a space, we write $X_+$ for the pointed space $(X, x_0)$ (so $X_+$ is the space $X$ with a disjoint basepoint added).

For a pair $(X, A)$, we can form the quotient space $X/A$, and take as a basepoint the point corresponding to $A$. In the case $A = \emptyset$, we set $X/\emptyset = X_+$.

Obs 2: A map $(X, A) \rightarrow (Y, B)$ induces a pointed map $X/A \rightarrow Y/B$. The maps induced by homotopic maps of pairs are pointed homotopic.

Reduced and relative $K$-theory

Def 3: $Sp. X = (X, x_0)$ is a pointed compact Hausdorff space. We define the reduced $K$-$\text{theory}$ of $X$ to be

$$\hat{K}_F(X) := \text{Ker}(K_F(X) \xrightarrow{i^*} K_F(x_0))$$

where $i$ denotes the inclusion $i: \{x_0\} \hookrightarrow X$. 
Equivalently,
\[ \tilde{\mathcal{K}}_{\text{IF}}(\mathcal{X}) = \ker (\mathcal{K}_{\text{IF}}(\mathcal{X}) \xrightarrow{d} \mathcal{Z}) \]
where \(d\) is the map given by \(d([\xi]) = \dim(\xi_0)\)
for v.b.'s \(\xi\).

**Def 4:** \(\mathcal{S}_{\mathcal{P}}\), \(\mathcal{X}\) is a compact Hausdorff space and
\(\mathcal{A} \subset \mathcal{X}\) is a closed subspace (i.e. \((\mathcal{X}, \mathcal{A})\) is a compact pair). We define the \(\mathcal{K}_{\text{IF}}\)-theory
of the pair \((\mathcal{X}, \mathcal{A})\) (or the \(\mathcal{K}_{\text{IF}}\)-theory of \(\mathcal{X}\) relative to \(\mathcal{A}\)) to be
\[ \mathcal{K}_{\text{IF}}(\mathcal{X}/\mathcal{A}) := \tilde{\mathcal{K}}_{\text{IF}}(\mathcal{X}/\mathcal{A}) \]

**Pth 5:** Notice that \(\tilde{\mathcal{K}}_{\text{IF}}(\mathcal{X})\) inherits a multiplication
from \(\mathcal{K}(\mathcal{X})\), but in general this product
is non-unital (\(\tilde{\mathcal{K}}_{\text{IF}}(\mathcal{X})\) is a "ring without a unit").

We will now attempt to quantify the difference
between \(\mathcal{K}_{\text{IF}}(\mathcal{X})\) and \(\tilde{\mathcal{K}}_{\text{IF}}(\mathcal{X})\). The following
is a standard result concerning short exact
sequences.

**Lemma 6:** Suppose
\[ 0 \to A \to B \overset{p}{\to} C \to 0 \quad (1) \]
is a short exact sequence of abelian groups
(or \(\mathbb{R}\)-modules, ...). Then the following are equivalent:


(i) There is a homomorphism \( s : C \to B \) s.t. \( ps = id_C \)
(a section for \( p \))

(ii) There is a homomorphism \( r : B \to A \) s.t. \( ri = id_A \)
(a retraction for \( i \))

(iii) There is an isomorphism \( \Theta : B \to A \oplus C \) s.t. the diagram

\[
\begin{array}{ccc}
0 & \to & A \\
\downarrow & & \downarrow \Theta \\
\downarrow & & \downarrow \\
A \oplus C & \to & 0
\end{array}
\]

commutes, where \( i_A : A \to (a, 0) \) and \( p_C : (a, c) \to c \)
are the evident inclusion and projection maps.

\( pf : (iii) \Rightarrow (i) \& (ii) : \) Given \( \Theta \), we can define
\( s = \Theta^{-1} i_C \) and \( r = p_A \Theta \)
where \( i_C : C \to A \oplus C \),
\( c \mapsto (0, c) \) and \( p_A : A \oplus C \to A \), \( (a, c) \mapsto a \)
are the evident maps.

\( (i) \Rightarrow (ii) : \) Given \( s \), define a map \( \bar{r} : B \to B \) by setting \( \bar{r}(b) = b - sp(b) \).
Then for all \( b \in B \),
\[
\bar{r}(b) = p(b - sp(b)) = p(b) - p(s)p(b) = p(b) - p(b) = 0,
\]
so \( \bar{r} \) lifts to a map \( r : B \to A \) s.t. \( ir = \bar{r} \).

Now \( ir(a) = \bar{r}i(a) = i(a) - spi(a) = i(a) \)
for all \( a \in A \). Since \( i \) is a monomorphism, it follows that \( ri = id_A \).
(ii) = (i): Given \( r \), define a map \( \tilde{s} : B \to B \) by setting \( \tilde{s}(b) = b - ir(b) \). Then for all \( a \in A \),

\[ \tilde{s}(i(a)) = i(a) - ir(i(a)) = i(a) - i(a) = 0, \]

so \( \tilde{s} \) descends to a map \( s : C \to B \) s.t.

\[ \tilde{s} = sp. \]

Now

\[ ps p(b) = p \tilde{s}(b) = p(b) - pin(b) = p(b) \]

for all \( b \in B \). Since \( p \) is an epimorphism, it follows that \( ps = id_C \).

(i) \& (ii) \Rightarrow (iii): Define \( \Theta : B \to A \oplus C \) by

\[ \Theta(b) = (r(b), p(b)). \]

Then \( \Theta \) makes the diagram in (iii) commutative. If \( \Theta(b) = 0 \), then \( p(b) = 0 \) and \( r(b) = 0 \). From the former we get that \( b = i(a) \) for some \( a \in A \), and hence the latter gives \( a = ri(a) = 0 \). Thus \( b = i(a) = 0 \), and \( \Theta \) is a monomorphism. To see that \( \Theta \) is an epimorphism, \( sp. \)

Now

\[ \Theta(i(a - rs(c)) + s(c)) \]

\[ = (ri(a - rs(c)) + rs(c), pi(a - rs(c)) + ps(c)) \]

\[ = (a - rs(c) + rs(c), ps(c)) \]

\[ = (a, c). \]

The claim follows. \( \Box \)
Def 7: If $s$ or $r$ as in Lemma 6, the short exact sequence (1) is called split.

Now let $(\hat{X}, x_0)$ be a pointed compact Hausdorff space. The maps induced by the inclusion $i: \{x_0\} \to \hat{X}$ and the unique map $r: \hat{X} \to \{x_0\}$ satisfy $i^*r^* = id_{K_\hat{F}(x_0)}$, so we get a split short exact sequence

$$0 \to \hat{K}_\hat{F}(X) \to K_\hat{F}(X) \xrightarrow{i^*} K_\hat{F}(x_0) \to 0.$$ 

Since $K_\hat{F}(x_0) \cong \mathbb{Z}$, we get an isomorphism of abelian groups

$$K_\hat{F}(X) \cong \hat{K}_\hat{F}(X) \oplus \mathbb{Z}.$$ 

We now turn to the construction of induced maps. If $f: (\hat{X}, x_0) \to (\hat{Y}, y_0)$ is a map of compact pointed Hausdorff spaces, the map

$$f^*: K_\hat{F}(X) \to K_\hat{F}(Y)$$

restricts to give an induced map

$$f^*: \hat{K}_\hat{F}(X) \to \hat{K}_\hat{F}(Y).$$

Diagrammatically, we have the following commutative diagram with exact rows:

$$
\begin{array}{ccc}
0 & \to & \hat{K}_\hat{F}(X) \\
\downarrow & & \downarrow \\
K_\hat{F}(X) & \xrightarrow{i^*} & K_\hat{F}(Y) \\
\downarrow & & \downarrow \\
K_\hat{F}(x_0) & \to & 0
\end{array}$$

$$(2)$$
The map \( \varphi \) only depends on the based homotopy class of \( f \).

If \( f : (Y, A) \to (Z, B) \), we define the induced map

\[
f_* : K_{\text{fr}}(Y, B) \to K_{\text{fr}}(Z, A)
\]

(3)

To be the map

\[
f_* = \tilde{f}^* : \tilde{K}_{\text{fr}}(Y/B) \to \tilde{K}_{\text{fr}}(Z/A)
\]

where \( \tilde{f} : Z/A \to Y/B \) is the map defined by \( f \). The map (3) only depends on the homotopy class of \( f \) as a map \((Z, A) \to (Y, B)\).

Theorem 8 (Excision): \( S_0 \cdot (Y, A) \) is a compact pair, and let \( U \subset Y \) be an open subset s.t. \( U \subset A \). Then the inclusion \((Y \setminus U, A \setminus U) \to (Y, A)\) induces an isomorphism

\[
K_{\text{fr}}(Y, A) \xrightarrow{\cong} K_{\text{fr}}(Y \setminus U, A \setminus U).
\]

Proof: \((Y \setminus U, A \setminus U) \to (Y, A)\) induces a homomorphism

\[
(Y \setminus U)/(A \setminus U) \xrightarrow{\cong} Y/A.
\]
Theorem 9: From the commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \to & K_1^F(\Sigma^+) \\
\downarrow & & \downarrow \\
0 & \to & K_1^F(X) \\
\end{array}
\]

we get a natural isomorphism

\[
\tilde{K}_1^F(\Sigma^+) \cong K_1^F(X).
\]

From this isomorphism, we get a natural iso

\[
K_1^F(\Sigma^+; \phi) = \tilde{K}_1^F(\Sigma^+/\phi) = \tilde{K}_1^F(\Sigma^+) \cong K_1^F(X).
\]

(This is compatible with the idea that we should be able to treat a span \( \Sigma \) as a pair \( (\Sigma, \phi) \).)

The reduced K-groups have a nice interpretation in terms of stable equivalence classes of bundles.

Definition 10: Call u.v.'s \( \Sigma, \Sigma' \to \Sigma \) stably equivalent, \( \Sigma \sim \Sigma' \), if \( \exists m, n \geq 0 \) s.t. \( \Sigma \oplus \varepsilon_m \sim \Sigma' \oplus \varepsilon_n \).

It is easy to verify that \( \sim \) is an equivalence relation and that \( \sim \) defines an equivalence relation on \( \text{Vect}_F(\Sigma) \). Write \( [\Sigma] \) for the \( \sim \)-equivalence
class of $x$.

Notice that Whitney sum gives a well-defined addition on $\text{Vect}_F(\mathbb{R})/\mathbb{N}$ with neutral element $0 = [e^0]_S$. If $X$ is compact Hausdorff, then any v.s. $\xi \rightarrow X$ admits a complement $\xi$ with $\xi \oplus \xi \sim \mathbb{R}^n$ for some $n \geq 0$, and then

$$[\xi]_S + [\xi]_S = [\xi \oplus \xi]_S = [\mathbb{R}^n]_S = [e^n]_S = 0$$

in $\text{Vect}_F(\mathbb{R})/\mathbb{N}$, so that $[\xi]_S = -[\xi]_S$.

Thus, in this case, $\text{Vect}_F(\mathbb{R})/\mathbb{N}$ is an abelian group.

Prop II: $S_0((\mathbb{R}, x_0))$ is a pointed compact Hausdorff space. Then the map

$$\varphi: \text{Vect}_F(X)/\mathbb{N} \rightarrow \text{Top}_F(X)$$

$$[\xi]_S \rightarrow [\xi - [e^\dim(x_0)]]$$

is an isomorphism of abelian groups.

Sketch of $\varphi$: It is easy to check that $\varphi$ is a well-defined homomorphism. It follows from Rh XII.1 that $\varphi$ is an epimorphism, and from Rh XII.3 that $\varphi$ is a mono.
Constructions on pointed spaces

Def 19: Let \( X = (X, x_0) \) be a pointed space. We define the (reduced) cone on \( X \) to be the quotient

\[
CX = C(X, x_0) = \frac{X \times I}{X \times \{1\} \cup \{x_0\} \times I}
\]

equipped with the basepoint given by the collapsed subspace. (Thus the reduced cone is obtained from the unreduced cone by collapsing the line \( \{x_0\} \times I \) to a point.) The map

\[
\begin{array}{ccc}
X & \longrightarrow & CX \\
\uparrow & & \uparrow \\
X & \longrightarrow & [x, 0]
\end{array}
\]

is a pointed embedding, and we use it to identify \( X \) with a subspace of \( CX \). Picture:

![Diagram](image)

(The dashed line is collapsed to a point.)

We use the same notation for the reduced and unreduced cones, and rely on context to make it clear which one is intended.
Lemma 13: For any pointed space \((X, x_0)\), the cone \(C(X, x_0)\) deformation retracts onto its basepoint.

\[ \text{Pf: The map} \]
\[ C(X, x_0) \times I \longrightarrow C(X, x_0) \]
\[ ([x, s], t) \longmapsto [x, (1-t)s + t] \]
provides the deformation retraction. \(\square\)

Thus all reduced cones are pointed homotopy equivalent to the one-point space.