Reduction to finding $\alpha$

Recall that we are trying to prove

**Thm 1 (Bott periodicity)**: Let $X$ be a pointed compact Hausdorff space. Then the map

$$\mathbb{K}(X) \xrightarrow{\beta} \mathbb{K}(X \wedge S^2)$$

given by reduced external product with the Bott class $b = [H] - 1 \in \mathbb{K}(S^2)$ is an isomorphism.

Our strategy is to reduce **Thm 1** to

**Prop 2**: For every compact space $X$, there exists a map

$$\alpha: \mathbb{K}(X \times S^2) \longrightarrow \mathbb{K}(X)$$

5.1.

1) $\alpha$ is natural in $X$

2) $\alpha$ is $\mathbb{K}(X)$-linear

3) For $X = pt$ and $b \in \mathbb{K}(S^2)$ the Bott class, $\alpha(b) = 1 \in \mathbb{K}(pt)$.

Assuming that such an $\alpha$ exists, we constructed a reduced version

$$\alpha: \tilde{\mathbb{K}}(X \wedge S^2) \longrightarrow \tilde{\mathbb{K}}(X)$$
which we claim to be an inverse for $\beta$. We saw

**Lemma 3:** The composite

$$
\tilde{\mathcal{K}}(X) \xrightarrow{\beta} \tilde{\mathcal{K}}(X \wedge S^2) \xrightarrow{\alpha} \tilde{\mathcal{K}}(X)
$$

is the identity map. \( \square \)

We also showed that the diagram

$$
\begin{array}{ccc}
\tilde{\mathcal{K}}(X \wedge S^2) & \xrightarrow{\beta} & \tilde{\mathcal{K}}(X \wedge S^2 \wedge S^2) \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
\tilde{\mathcal{K}}(X) & \xrightarrow{\beta} & \tilde{\mathcal{K}}(X \wedge S^2)
\end{array}
$$

commutes, where $T: S^2 \wedge S^2 \to S^2 \wedge S^2$ is the map $T(x \wedge y) = y \wedge x$.

**Lemma 4:** The map

$$
\tilde{\mathcal{K}}(X \wedge S^2 \wedge S^2) \xrightarrow{(\text{id}_X \wedge T)^*} \tilde{\mathcal{K}}(X \wedge S^2 \wedge S^2)
$$

is the identity.

**Pf:** Regard $S^2 \wedge S^2 \cong S^4$ as the one-point compactification of $\mathbb{R}^4$. Then $T$ is induced by the linear map

$$
T: \mathbb{R}^4 \to \mathbb{R}^4, \quad (x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_1, x_2).
$$

Since $\det(T) = 1$, we can connect $T$ to $\text{id}_{\mathbb{R}^4}$ by a path in $\text{GL}_4(\mathbb{R})$. This yields a homotopy from $T$ to $\text{id}_{S^2 \wedge S^2}$, and hence from $\text{id}_X \wedge T$ to $\text{id}_{X \wedge S^2 \wedge S^2}$. \( \square \)
We conclude that in the diagram

\[ \beta \alpha = \alpha (\text{id}_{X^2 T})^* \beta = \alpha \beta = \text{id} \tilde{\kappa}(X^2 S) \]

Diagram \quad \text{Lemma 3} \quad \text{Lemma 4}

Conclusion: \( \alpha : \tilde{\kappa}(X^2 S) \to \tilde{\kappa}(X) \) and \( \beta : \tilde{\kappa}(X) \to \tilde{\kappa}(X^2 S) \)

are inverse isomorphisms, so Prop 2 implies Thm 1.

We are left with the task of constructing the maps \( \alpha \) as in Prop 2. We will do so by considering successively more general clutching functions.

**Linear clutching functions**

Let \( S \to X \) be a complex u.c.1, and let \( \rho : \pi^* S \to \pi^* S \) be a linear clutching function, i.e.

\[ \rho(x, z) = a(x) z + b(x) \]

for some endomorphisms \( a, b : S \to S \).

(As before, \( \alpha : X \times S^1 \to X \) is the projection.)

We will use \( \rho \) to construct a decomposition

\[ S = (S, \rho)_+ \oplus (S, \rho)_- \]

of \( S \). To do so, we will construct a projection operator on \( S \).
Def 5: A projection operator on a v.b. $\mathcal{E} \rightarrow \mathcal{X}$ is a v.b. endomorphism $Q : \mathcal{E} \rightarrow \mathcal{E}$ s.t. $Q^2 = Q$.

Write $Q_x : \mathcal{E}_x \rightarrow \mathcal{E}_x$ for the restriction of $Q$. Notice that for each $x \in \mathcal{X}$, $\text{Im}(Q_x) \cap \text{Ker}(Q_x) = 0$, so $\mathcal{E}_x$ decomposes as $\mathcal{E}_x = \text{Im}(Q_x) \oplus \text{Ker}(Q_x)$.

We would like to promote this to a decomposition of vector bundles. For this, we need

Prop 6: Any projection operator $Q$ on a v.b. $\mathcal{E} \rightarrow \mathcal{X}$ is of locally constant rank (i.e. $\text{rank}(Q_x : \mathcal{E}_x \rightarrow \mathcal{E}_x)$ is a locally constant function of $x \in \mathcal{X}$).

Pf: Let $x \in \mathcal{X}$. Use a local trivialization of $\mathcal{E}$ near $x$ to construct sections $s_1, \ldots, s_n$ of $\mathcal{E}$ defined in a neighborhood of $x$ such that $s_1(x), \ldots, s_r(x)$ are a basis of $\text{Im}(Q_x)$ and $s_{r+1}(x), \ldots, s_n(x)$ are a basis of $\text{Ker}(Q_x)$.

Then for $y$ near $x$, the vectors

$$Qs_1(y), \ldots, Qs_r(y), (1 - Q)s_{r+1}(y), \ldots, (1 - Q)s_n(y) \in \mathcal{E}_y$$

form a basis of $\mathcal{E}_y$ (since they do at $y = x$). Since $Qs_1(y), \ldots, Qs_r(y) \in \text{Im} Q_y$ and $(1 - Q)s_{r+1}(y), \ldots, (1 - Q)s_n(y) \in \text{Ker} Q_y$, it follows that $\dim(\text{Im} Q_y) + \dim(\text{Ker} Q_y) = \dim(\mathcal{E}_y) = n$, it follows that $\dim(\text{Im} Q_y) \leq n - r$. Since $\dim(\text{Im} Q_y) + \dim(\text{Ker} Q_y)$

$$\geq \dim(\mathcal{E}_y) = n,$$

it follows that $\text{rank } Q_y = \dim(\text{Im} Q_y) = r.$
By the generalization of the construction of kernel and image of a constant-rank map of v.b.'s to the context of v.b.'s with varying fibre dimension, we obtain v.b.'s

\[ \text{Im} Q \to \mathcal{F} \text{ and } \text{Ker} Q \to \mathcal{F} \]

with fibres \( (\text{Im} Q)_x = \text{Im}(Q_x) \) and \( (\text{Ker} Q)_x = \text{Ker}(Q_x) \). Now

\[ \mathcal{E} = \text{Im} Q \oplus \text{Ker} Q \]

as vector bundles.

Let us now define a projection operator \( Q_p : \mathcal{E} \to \mathcal{E} \) associated to the linear clustering function

\[ \rho(x, z) = a(x)z + b(x) \]

as follows:

\[ Q_p(x) = \frac{1}{2 \pi i} \sum_{|z| = 1} \left[ a(x)z + b(x) \right]^{-1} a(x) dz \]

**Lemma 7:** \( Q_p : \mathcal{E} \to \mathcal{E} \) is a projection operator.

**Proof:** First notice that for \( z \neq w \), we have

\[
\left( \frac{(az+b)^{-1}}{w-z} + \frac{(aw+b)^{-1}}{z-w} \right) = \frac{1}{(az+b)^{-1} \frac{az+b}{z-w} (aw+b)^{-1}}
\]

\[
= (az+b)^{-1} a (aw+b)^{-1}
\]
Since \( a(x)z + b(x) \) is invertible for all \( x \in \mathbb{C} \), 
\( |z| = 1 \), we can find an \( \epsilon > 0 \) s.t. it is invertible for all \( x \in \mathbb{C} \), 
\( 1 - \epsilon < |z| < 1 + \epsilon \). Now the Cauchy integral theorem implies that

\[
Q_p(x) = \frac{1}{2\pi i} \int_{|z|=1} \left[ a(x)z + b(x) \right]^{-1} a(x) \, dz
\]

for any \( 1 - \epsilon < |z| < 1 + \epsilon \). Let \( 1 - \epsilon < r < R < 1 + \epsilon \).

Then

\[
Q_p = \frac{1}{(2\pi i)^2} \int_{|z|=r} \int_{|z|=R} \frac{(a_2 + b)^{-1}}{a(z + b)^{-1}} a \, dz \, dw
\]

Linearly

\[
= \frac{1}{(2\pi i)^2} \int_{|w|=r} \int_{|z|=R} \frac{(a_2 + b)^{-1}}{w-z} a \, dz \, dw
\]

By (x)

\[
= \frac{1}{(2\pi i)^2} \int_{|w|=r} \int_{|z|=R} \frac{(a_2 + b)^{-1}}{w-z} a \, dz \, dw
\]

Fubini, Cauchy integral formula

\[
= \int_{|w|=r} \int_{|z|=R} \frac{(a_2 + b)^{-1}}{w-z} a \, dw \, dz
\]

\[
+ \frac{1}{2\pi i} \int_{|w|=r} \left( a_2 + b \right)^{-1} a \, dw
\]

Therefore

\[
\frac{1}{2\pi i} \int_{|z|=1} 0 \, dz + Q_p = Q_p. \quad \square
\]

Cauchy integral theorem
Corollary: The u.b. $S \rightarrow \mathbb{A}$ decomposes as a direct sum

$$S = (S, \rho)_+ \oplus (S, \rho)_-$$

where $(S, \rho)_+ = \text{Im} \rho_+$ and $(S, \rho)_- = \text{Ker} \rho_-$.

Proposition: We have

(i) $(S, 1)_+ = e^0_A$ and $(S, 2)_+ = S$.

(ii) $(S, \rho_0)_+ \cong (S, \rho_1)_+$ if $\rho_0, \rho_1 : Y \rightarrow \mathbb{A}$ are homotopic through linear clutching functions.

(iii) $(S \oplus S_1, \rho_1 \oplus \rho_2)_+ \cong (S_1, \rho_1)_+ \oplus (S_2, \rho_2)_+$

(iv) $(S \otimes S_1, \text{id}_S \otimes \rho)_+ \cong S \otimes (S, \rho)_+$ for any complex u.b. $S \rightarrow \mathbb{A}$.

(Here the $S$'s are complex u.b.'s over $\mathbb{A}$ and $\rho$'s are linear clutching functions.)

Proof: (i):

$$Q_1 = \frac{1}{2\pi i} \int_{121=1} \text{id}_S \, d2 = 0 \quad \text{(by Cauchy integral theorem)}$$

$$Q_2 = \frac{1}{2\pi i} \int_{121=1} \text{id}_S \, d2 = \text{id}_S \quad \text{(by Cauchy integral formula or direct computation)}$$

(ii): A homotopy from $\rho_0$ to $\rho_1$ through linear clutching functions gives a linear clutching
function $P$ for $S \times I \rightarrow \Sigma \times I$, and $(S, \rho_0)_+$ and $(S, \rho_1)_+$ are the two ends of the u.b.

$$(S \times I, P)_+ \rightarrow \Sigma \times I.$$ 

(iii) & (iv) are immediate from the construction. $\Box$