Polynomial clutching functions

Recall: We are trying to prove Bott periodicity, and we have reduced the task to constructing maps

\[ \alpha: K(X \times S^2) \to K(X) \]

So that

1) \( \alpha \) is natural in \( X \)
2) \( \alpha \) is \( K(X) \)-linear
3) \( \alpha(b) = 1 \in K(pt) \), where \( b = [H]-1 \in K(S^2) \) is the Bott class.

Last time, given a complex u.b. \( S \to X \) and a linear clutching function \( \rho: \pi^*S \to \pi^*S \) \( (\pi: X \times S^1 \to X \text{ the projection}) \), we constructed a direct sum decomposition

\[ S = (S, \rho)_+ \oplus (S, \rho)_- \]

of u.b.'s over \( X \).

Suppose now \( \rho(x, z) = \sum_{k=0}^{n} a_k(x) z^k: \pi^*S \to \pi^*S \) is a polynomial clutching function of degree \( \leq n \).

We set

\[ L^n(S) = \bigoplus_{n+1} S \to X \]

and define a linear clutching function.
\[ L_n(p) : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

by

\[
L_n(p) = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\
-\bar{z} & 1 & 0 & \cdots & 0 & 0 \\
0 & -\bar{z} & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & -\bar{z} & 1 \\
\end{pmatrix}
\]

(Notice that then

\[
L_n(p) = \begin{pmatrix}
1 & g_1 & g_2 & \cdots & g_n \\
-\bar{z} & 1 & 0 & \cdots & 0 \\
0 & -\bar{z} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & -\bar{z} \\
\end{pmatrix}
\]

where the polynomials \( g_i \) are defined inductively

by \( g_0 = p, \; g_{r+1}(z) = (g_r(z) - g_r(0))/\bar{z} \); it follows that \( L_n(p) \) indeed is an isomorphism.

We set

\[ L_n(s, p)_+ := (L_n(s), L_n(p))_+ \rightarrow \mathbb{R} \]

Prop 1: We have

(i) \( L_{n+1}(s, p)_+ \cong L_n(s, p)_+ \) and

\( L_{n+1}(s, zp)_+ \cong L_n(s, p)_+ \oplus 5 \)

if \( p \) is polynomial of degree \( \leq n \).
(ii) \( L^n(S, p_0) \approx L^n(S, p_1) \) if \( p_0 \) and \( p_1 \) are homotopic through polynomial clutching functions of degree \( \leq n \).

(iii) \( L^n(S \oplus S_2, p \oplus p_2) \approx L^n(S, p) \oplus L^n(S_2, p_2) \)

(\( p_1, p_2 \) polynomial of degree \( \leq n \))

(iv) \( L^n(\eta \circ \xi, \text{id}_{K(S)} \circ p) \approx \eta \circ L^n(S, p) \)

(\( p \) polynomial of degree \( \leq n \), \( \eta : \text{Id} \to \Delta \) a complex v.b.)

\[ \textbf{Pf. (i): Notice that} \]

\[
\begin{pmatrix}
L^n(p) & 0 \\
0 & -Z^n(p)
\end{pmatrix}, \quad 0 \leq t \leq 1
\]

gives a homotopy from \( L^n(p) \oplus 1 \) to \( L^m(p) \)
through linear clutching functions. On the other hand, for \( \alpha : I \to \text{Aut}(S) \) a path connecting \( I_2 \) to \( (0, -1) \), the product

\[
\begin{pmatrix}
\alpha(t) & 0 \\
0 & I_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & a_0 & a_1 & \cdots & a_n \\
-2 & 1 & -t & \cdots & -t \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-2 & \cdots & \cdots & \cdots & 1
\end{pmatrix}, \quad 0 \leq t \leq 1
\]

gives a homotopy from \( L^{m_1}(2p) \) to \( 2 \oplus L^n(p) \)
through linear clutching functions. The claim now follows from Prop. XIX.9 (ii), (iii) and (i).
(ii): A homotopy from \( p_0 \) to \( p_1 \) through polynomial clutchering functions yields a homotopy from \( L^n(p_0) \) to \( L^n(p_1) \) through linear clutchering functions. The claim now follows from Prop. \textsc{xix}.9.(ii).

(iii) and (iv) are immediate from the construction and Prop. \textsc{xix}.9.(iii),(iv). \( \square \)

General clutchering functions

Given a complex v.b. \( S \longrightarrow X \) and a laurent polynomial clutchering function

\[
\lambda: \mathbb{C}^k \cong \mathbb{C}^k
\]

for \( n \) large enough, \( z^n \lambda \) is a polynomial clutchering function of degree \( \leq 2n \). We define

\[
\alpha(S, \lambda) := \chi S - \alpha_{2n}(S, z^n \lambda) \in \chi(X)
\]

By Prop. 1.(i), \( \alpha(S, \lambda) \) is independent
of the choice of \( n \gg 0 \).

By Prop. 1.(ii), we have \( \alpha(S, l_0) = \alpha(S, l_1) \)
if \( l_0 \) and \( l_1 \) are homotopic through laurent polynomial clutchering functions. By Cor. \textsc{xviii}.3.(ii),
it follows that more generally, \( \alpha(S, l_0) = \alpha(S, l_1) \)
whenever \( l_0, l_1 \) are laurent polynomial clutchering functions homotopic through clutchering functions.
For an arbitrary clutching function
\[ u : \pi_5^s \to \pi_5^s \]
we now obtain a well-defined element

\[ \alpha(s,u) \in \mathcal{K}(X) \]

by setting \( \alpha(s,u) = \alpha(s, l) \), where \( l \) is a
Laurent polynomial clutching function homotopic to \( u \) through clutching functions.
(Such an \( l \) exists by Cor. XVIII.3.(i).)

**Prop 2**: We have

(i) \( \alpha(s, u_0) = \alpha(s, u_1) \in \mathcal{K}(X) \) if \( u_0 \) and \( u_1 \)
are homotopic through clutching functions \( \pi^s_5 \cong \pi^s_5 \).

(ii) \( \alpha(s, \psi_2, u_1 \circ u_2) = \alpha(s, u_1) + \alpha(s, u_2) \in \mathcal{K}(X) \)
\( (s, \psi \to X \text{ complex u.b.'s, } u_i : \pi^s_5 \to \pi^s_5, \ i = 1, 2, \)
clutching functions).

(iii) \( \alpha(\eta \circ s, \text{id}_{\eta} \circ u) = \eta \alpha(s, u) \in \mathcal{K}(X) \)
\( (\eta, \psi \to X \text{ complex u.b.'s, } u : \pi^s_5 \to \pi^s_5 \text{ clutching function}) \).

**Pf:** (i) is immediate from the definition.

(ii) and (iii) follow from Prop. 1. (iii) and (iv), respectively. \( \Box \)
Construction of $\alpha: K(\mathbb{X} \times S^2) \to K(\mathbb{X})$

For a complex v.b. $\mathbb{S} \to \mathbb{X} \times S^2$, set $\mathbb{S} := \mathbb{S} \setminus \mathbb{X}$ (where we identify $\mathbb{X}$ with $\mathbb{X} \times \{1\} \subset \mathbb{X} \times S^2$), and define

$$\alpha(\mathbb{S}) := \alpha(\mathbb{S}, u) \in K(\mathbb{X})$$

where $u: \overline{\mathbb{X}} \to \overline{\mathbb{X}^1}$ is the clutching function (unique up to homotopy) afforded by Prop. XVIII.4. (Then $\mathbb{S} \cong \mathbb{X} \cup_{\partial \mathbb{X}} \mathbb{S}$.) By Prop. 2.2(i), $\alpha(\mathbb{S})$ is well defined. Prop. 2.2(ii) and (iii) imply

**Prop 3**: We have

(i) $\alpha(\mathbb{S}_1 \oplus \mathbb{S}_2) = \alpha(\mathbb{S}_1) + \alpha(\mathbb{S}_2) \in K(\mathbb{X})$

(\(\mathbb{S}_1, \mathbb{S}_2 \to \mathbb{X} \times S^2\) complex v.b.'s).

(ii) $\alpha(\varphi^*(\eta) \otimes \mathbb{S}) = \eta \alpha(\mathbb{S})$

($\eta \to \mathbb{X}$, $\mathbb{S} \to \mathbb{X} \times S^2$ complex v.b.'s, $\varphi^*: \mathbb{X} \times S^2 \to \mathbb{X}$ the projection). $\square$

By Prop. 3.(i), the map

$$\text{Vect}_{\mathbb{X}}(\mathbb{X} \times S^2) \xrightarrow{\alpha} K(\mathbb{X})$$

induces a homomorphism

$$K(\mathbb{X} \times S^2) \xrightarrow{\alpha} K(\mathbb{X})$$

of abelian groups, and by Prop. 3.(ii), the
induced map is $K(\mathbb{Z})$-linear. Tracing through the construction, we see that $\alpha$ is natural in $\mathbb{Z}$. It remains to show that $\alpha(b) = 1 \in K(pt)$ for $b = [H] - 1 \in K(S^2)$. This follows from

**Lemma 4:**

(i) $\alpha(H) = 1 \in K(pt)$

(ii) $\alpha(\varepsilon_{S^2}^1) = 0 \in K(pt)$.

**Pf:** (i): From Lecture 11 it follows that as the clutching function $u : \pi^*H \to \pi^*H$, we may pick $u = \varepsilon^{-1}$. Thus,

$$\alpha(H) = \alpha(H_1, \varepsilon^{-1}) = H_1 - \alpha^2(H_1, 1)_+ = 1 - (\alpha^2(H_1), \alpha^2(1))_+ \in K(pt).$$

Here

$$\alpha^2(1) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Through linear clutching function

Thus by Prop. XIX.9. (ii), (iii) and (i) we have

$$(\alpha^2(H_1), \alpha^2(1))_+ = 0 \in K(pt),$$

and the claim follows.

(ii): As the clutching function $u : \pi^*E_{pt} \to \pi^*E_{pt}$ we may pick the identity map $1$. Thus

$$\alpha(\varepsilon_{S^2}^1) = \alpha(\varepsilon_{pt}^1, 1) = -\alpha^0(\varepsilon_{pt}^1, 1)_+ = -\varepsilon_{pt}^1 = 0 \in K(pt). \square$$

Prop. XIX.9.(i)

This concludes the proof of Bott periodicity.