The splitting principle (cont.)

Recall: We want to prove

**Thm. 1 (Splitting principle):** $Sp : X$ is a compact Hausdorff space and let $g : X \to X$ be a complex v.f. Then there exist a compact Hausdorff space $Y$ and a map $p : Y \to X$ s.t.

$$p^* : K^*(X) \to K^*(Y)$$

is injective and $p^* : Y \to X$ splits as a direct sum of line bundles.

Last time, we proved

**Thm. 2 (Leray-Hirsch):** Let $p : E \to B$ be a fibre bundle with $E, B$ compact Hausdorff and fibre $F$ a finite cell complex with only even-dimensional cells. Suppose there exist classes $c_1, \ldots, c_k \in K(E)$ which restrict to a basis for $K(E_b)$ for each fibre $E_b = p^{-1}(b)$ of $p : E \to B$. Then $K^*(E)$ is a free $K^*(B)$-module with basis $\{c_1, \ldots, c_k\}$.

The main remaining ingredient in the proof of Thm. 1 is the computation of $K(CP^n)$.

Recall: $CP^n$ is a finite cell complex with a single cell of dimension $2k$ for all $0 \leq k \leq n$. Thus $K^i(CP^n) = 0$ and $K^0(CP^n)$ is free abelian of rank $n+1$ (Cor. XXII.2).
Thm 3: \( K(\mathbb{C}P^n) = \mathbb{Z}[L]/(L-1)^{n+1} \) where
\( L \to \mathbb{C}P^n \) is the dual of the canonical line bundle \( \mathcal{O}(1) \to \mathbb{C}P^n \).

Thus in particular \( K(\mathbb{C}P^n) \) has additive basis \( 1, L, \ldots, L^n \).

Pf of Thm 3: By induction on \( n \). \( \mathbb{C}P^0 = \mathbb{P}^1 \), so the claim holds for \( n = 0 \). For \( n > 0 \), the long exact sequence of \( (\mathbb{C}P^n, \mathbb{C}P^{n-1}) \) gives a short exact sequence
\[
0 \to K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \to K(\mathbb{C}P^n) \xrightarrow{p} K(\mathbb{C}P^{n-1}) \to 0.
\]

The claim will follow if we can show
\((An)\quad \ker(p) \) is generated by \( (L-1)^n \).

Then induction and the above sequence imply that \( 1, L-1, \ldots, (L-1)^n \) is an additive basis for \( K(\mathbb{C}P^n) \). Moreover, by \((An,1)\) we have
\( (L-1)^{n+1} = 0 \) in \( K(\mathbb{C}P^n) \), so \( K(\mathbb{C}P^n) = \mathbb{Z}[L]/(L-1)^{n+1} \) as claimed.

It remains to show \((An)\) for all \( n > 0 \).
Notice that \( K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \cong \mathbb{C}(s^n) \), we know that \( \mathbb{C}(s^n) \) is generated by \( (H-1)^n \).

The basic idea is to relate the internal
product
\[ \mathcal{N}(C^p) \otimes - \otimes \mathcal{N}(C^p) \longrightarrow \mathcal{N}(C^p) \]

by the external product
\[ \mathcal{N}(S^2) \otimes - \otimes \mathcal{N}(S^2) \longrightarrow \mathcal{N}(S^2) \]

Instead of interpreting \( S^p \) as the orbit space \( S^{n+1}/\mathbb{Z} \) where \( S^1 \subset \mathbb{C} \) acts on \( S^{n+1} \subset \mathbb{C}^{n+1} \) by scalar multiplication, we may interpret \( S^p \) as the orbit space
\[ C^p = S(D_1^2 \times \cdots \times D_{n+1}^2) / S \]

where \( D_i^2 \) is the unit disk in the \( i \)-th coordinate in \( C^{n+1} \), and the action is again by scalar multiplication. We have a decomposition
\[ \mathcal{N}(D_1^2 \times \cdots \times D_{n+1}^2) = U_{i=1}^{n+1} D_i^2 \times \cdots \times D_i^2 \times D_{n+1}^2 \]

into \( S^1 \)-invariant subspaces. The quotient of the \( i \)-th member of the union by the \( S^1 \)-action is a subspace \( C_i \subset C^p \) homeomorphic to \( D_1^2 \times \cdots \times D_{n+1}^2 \) with \( D_i^2 \) omitted. Explicitly, a homeomorphism is given by
\[ \psi_i : D_1^2 \times \cdots \times \hat{D_i}^2 \times \cdots \times D_{n+1}^2 \longrightarrow C_i \]
\[ (z_1, \ldots, \hat{z_i}, \ldots, z_{n+1}) \longmapsto [z_1, \ldots, 1, \ldots, z_{n+1}] \]

where \( \hat{\cdot} \) indicates omission. Thus we have a decomposition
\[ C^p = \bigcup_{i=1}^{n+1} C_i \]

with each \( C_i \) homeomorphic to \( D_1^2 \).
Identify $C_{n+1}$ and $D_1^2 \times \ldots \times D_n^2$ via $\varphi_{n+1}$

and let

\[ \Theta : C_{n+1} = D_1^2 \times \ldots \times D_i^2 \times \ldots \times D_n^2. \]

We now have the following commutative diagram (where the morphisms are induced by inclusions unless indicated otherwise):

\[
\begin{array}{ccc}
K(D_1^2, \mathbb{R}^2) \otimes \ldots \otimes K(D_n^2, \mathbb{R}^2) & \xrightarrow{\simeq} & K(D_1^2 \times \ldots \times D_n^2, \mathbb{R}^2) \\
1 \simeq & & \simeq \varphi_{n+1}^* \\
K(C_{n+1}, \mathbb{R}^2, C_{n+1}) \otimes \ldots \otimes K(C_{n+1}, \mathbb{R}^2, C_{n+1}) & \to & K(C_{n+1}, \mathbb{R}^2, C_{n+1}) \\
\uparrow & & \uparrow \\
K(CP^n, C_1) \otimes \ldots \otimes K(CP^n, C_n) & \to & K(CP^n, C_1 \cup \ldots \cup C_n) \cong \\
\downarrow & & \downarrow \\
K(CP^n) \otimes \ldots \otimes K(CP^n) & \to & K(CP^n)
\end{array}
\]

(*)

Several of the maps in the diagram are isomorphisms. This can be justified as follows:

1. \( (D_1^2, \mathbb{R}^2) \to (C_{n+1}, \mathbb{R}^2, C_{n+1}) = D_1^2 \times \ldots \times (D_i^2, \mathbb{R}^2, C_{n+1}) \times \ldots \times D_n^2 \)

   is a homotopy equivalence of pairs (with homotopy inverse given by the projection).

2. By comparison with \( K(S^2) \otimes \ldots \otimes K(S^2) \xrightarrow{\simeq} K(S^{2n}) \).

3. \( \varphi_{n+1} \) is a homomorphism.

4. \( C_{n+1} \to CP^n \) induces a homomorphism

   \[ C_{n+1} / \partial C_{n+1} \xrightarrow{\simeq} CP^n / C_1 \cup \ldots \cup C_n \]
6. Notice that $\mathbb{C}P^{n-1} \cong \{ [z_1, \ldots, z_n : 0] \in \mathbb{C}P^n \}$ does not intersect $C_i + 1$, so $\mathbb{C}P^{n-1} \cup C_i \cup \cdots \cup C_n$. It is easy to show that the quotient map

$$\mathbb{C}P^n / \mathbb{C}P^{n-1} \rightarrow \mathbb{C}P^n / C_i \cup \cdots \cup C_n$$

is a pointed homotopy equivalence. (We have a commutative square

$$\begin{array}{ccc}
[\nu_1 : 1 : \nu_2] & \mathbb{C}P^n / \mathbb{C}P^{n-1} & \mathbb{C}P^n / C_i \cup \cdots \cup C_n \\
\downarrow & \uparrow \cong \uparrow & \uparrow \\
[\nu] & D^{2n} / \partial D^{2n} & D^2_i \times \cdots \times D^2_n / \partial (D^2_i \times \cdots \times D^2_n)
\end{array}$$

where $\alpha$ is induced by

$$D^{2n} \rightarrow D^2_i \times \cdots \times D^2_n$$

$$\nu \mapsto \begin{cases} \nu / |1-\nu| & \text{if } \|\nu\|_\infty \leq 1-|\nu| \\ \nu / \|\nu\|_\infty & \text{otherwise} \end{cases}$$

( $\| (z_1, \ldots, z_n) \|_\infty = \max_{1 \leq i \leq n} |z_i| $). Now check that

$$D^2_i \times \cdots \times D^2_n \rightarrow D^{2n}$$

$$\nu \mapsto \frac{\nu}{\|\nu\|_\infty}$$

induces a homotopy inverse for $\alpha$.)

Since $C_i$ is contractible, the map

$$\pi_0(\mathbb{C}P^n, C_i) \xrightarrow{\text{ind}^*} \pi_0(\mathbb{C}P^n)$$

gives an isomorphism

$$\pi_0(\mathbb{C}P^n, C_i) \cong \tilde{\pi}_0(\mathbb{C}P^n).$$
It follows that $L^{-1} \in \tilde{K}(C \mathbb{P}^n)$ lifts to a unique class $x; \in K(C \mathbb{P}^n, C;)$. Interpret

$$C \mathbb{P}^1 = D^2 \cup D^2 \infty$$

where

$$D^2 = \{[z: 1] \mid |z| \leq 1\}$$

$$D^2 \infty = \{[1: z] \mid |z| \leq 1\}.$$

Now the commutative diagram

$$\begin{array}{ccc}
[2: 1] & \rightarrow & 12 \\
\downarrow & & \downarrow \\
[3: 2] & \rightarrow & (C \mathbb{P}^1, D^2) \\
\downarrow & & \downarrow \\
(CP^n, C;) & \rightarrow & (C \mathbb{P}^n, C;) \\
\end{array}$$

gives a commutative diagram excision

$$\begin{array}{ccc}
H-1 \tilde{K}(C \mathbb{P}^1) & \leftarrow & \tilde{K}(C \mathbb{P}^1, D^2) \rightarrow \tilde{K}(D^2, D^2) \\
\uparrow & & \uparrow \\
\tilde{K}(C \mathbb{P}^n) & \leftarrow & \tilde{K}(C \mathbb{P}^n, C;) \\
\end{array}$$

We deduce that in (5), the element $x; \in K(C \mathbb{P}^n, C;) \mapsto$ the generator in $K(D^2, D^2)$, (An) now follows from diagram (8). Notice that $\tilde{K}(\mathbb{P}, C)$ is the image of the map

$$K(C \mathbb{P}^n, C^{-1}) \rightarrow K(C \mathbb{P}^n).$$
Recall that if $\mathcal{X} \rightarrow X$ is a complex vector bundle of constant dimension $n$, we have a fibre bundle $P(\mathcal{X}) \rightarrow X$ with fibre $P(\mathcal{X})_x$ over $x \in X$; the span $P(\mathcal{X})_x \cong \mathbb{C}^{n-1}$ of lines in $\mathcal{X}_x$. Over $P(\mathcal{X})$, we have a canonical line bundle

\[ L^* = \{ (l, v) \in P(\mathcal{X}) \times \mathbb{C} \mid v \in l \} \rightarrow P(\mathcal{X}) \\
(l, v) \mapsto l \]

with fibre $L^*_x \cong l$. Write $L$ for the dual of $L^*$. By Thm 3, the elements

\[ 1, L, \ldots, L^{n-1} \in K(P(\mathcal{X})) \]

restrict to a basis of $K(P(\mathcal{X})_x) \cong \mathbb{C}^{n-1}$ for each $x \in X$, so by Thm 2, $K^*(P(\mathcal{X}))$ is a free $K^*(\mathcal{X})$-module with basis $\{1, L, \ldots, L^{n-1}\}$.

Proof of Thm 1: Consider first the case where $\mathcal{X}$ is of constant dimension $n$. If $n \geq 0$, we may take $\mathcal{Y} = \overline{\mathcal{X}}$, $\mathcal{P} = \id_{\mathcal{X}}$. So $n > 0$. By the above discussion, since $1 \in K(P(\mathcal{X}))$ is among the basis elements, the map

\[ K(\mathcal{X}) \xrightarrow{K^*} K(P(\mathcal{X})) \]

induced by the projection $\overline{\pi}: P(\mathcal{X}) \rightarrow \overline{\mathcal{X}}$ is a monomorphism. Moreover, the pullback $K^*(\mathcal{Y})$ contains the canonical line bundle $L^*$ as a subbundle; the embedding $L^* \hookrightarrow K^*(\mathcal{X})$ is induced by
Thus $\tau^\cdot(\mathfrak{s})$ splits as a sum

$$\tau^\cdot(\mathfrak{s}) = L^* \oplus \mathfrak{s}'$$

for some $(n-1)$-dimensional v.b. $\mathfrak{s}' \to P(\mathfrak{s})$.

The claim now follows by induction. In the general case, $\mathfrak{s}$ splits as a disjoint union

$$\mathfrak{s} = \mathfrak{s}_1 \amalg \cdots \amalg \mathfrak{s}_k$$

of subspaces s.t. $\mathfrak{s}_i | \mathfrak{s}$; has constant dimension for each $i=1, \ldots, k$. The claim follows by considering each $\mathfrak{s}_i | \mathfrak{s}$ separately. $\square$

Let $\mathfrak{s} \to \mathfrak{X}$ be an $n$-dimensional complex v.b. We saw earlier that $U^*(P(\mathfrak{s}))$ is a free $U^*(\mathfrak{X})$-module with basis $\{1, L, \ldots, L^{n-1}\}$, let us now determine the multiplicative structure of $U^*(P(\mathfrak{s}))$. To do so, it is enough to express $L^n$ in terms of $1, L, \ldots, L^{n-1}$. The pullback of $\mathfrak{s}$ over $P(\mathfrak{s})$ splits as a sum $L^* \oplus \mathfrak{s}'$ with $\mathfrak{s}'$ $(n-1)$-dimensional. Continuing to write $\mathfrak{s}$ for the pullback, we have

$$\lambda^\cdot(\mathfrak{s}) = \lambda^\cdot(L^*) \lambda^\cdot(\mathfrak{s}')$$

so that

$$\lambda^\cdot(\mathfrak{s}') = \lambda^\cdot(\mathfrak{s}) \lambda^\cdot(L^*)^{-1} = \lambda^\cdot(\mathfrak{s})(1 + L^*)^{-1}$$

$$= \lambda^\cdot(\mathfrak{s}) \left( \sum_{i=0}^{\infty} (-1)^i (L^*)^i \right).$$
Since $\xi'$ is $(n-1)$-dimensional, $\lambda^n(\xi') = 0$.

Comparing the coefficients of $t^n$, we get that

$$0 = \sum_{i=0}^{n} (-1)^i \lambda^i(\xi) (L^*)^{n-i}$$

$$= \sum_{i=0}^{n} (-1)^i \lambda^i(\xi) L^{i-n}$$

in $K(P(\xi))$. Multiplying by $L^n$ now gives the identity

$$\sum_{i=0}^{n} (-1)^i \lambda^i(\xi) L^i = 0.$$

Notice that the coefficient $(-1)^n \lambda^n(\xi)$ of $L^n$ on the left hand side is a unit, so this allows us to write $L^n$ as a linear combination of lower powers of $L$. We conclude that

**Prop 4:** For $\xi \to \xi$ an $n$-dimensional complex

the ring $K^*(P(\xi))$ is isomorphic to

$K^*(\xi)[L]/(\sum_{i=0}^{n} (-1)^i \lambda^i(\xi) L^i). \ \square$